COMPLEX ANALYSIS COURSE NOTES

1. January 6

Let us quickly recall some basic properties of the real numbers, which we denote by $\mathbb{R}$.

Proposition 1.1. *Let a, b, c be real numbers.*

(1) $a + b$ and ab are also real numbers (closure).
(2) Addition is associative: $a + (b + c) = (a + b) + c$.
(3) Addition is commutative: $a + b = b + a$.
(4) There exists a real number, named 0, such that $a + 0 = a = 0 + a$.
(5) There exists a real number $-a$ such that $a + (-a) = 0 = (-a) + a$. For shorthand, we write $a + (-b)$ as $a - b$.
(6) Multiplication is associative: $a(bc) = (ab)c$.
(7) Multiplication is commutative: $ab = ba$.
(8) There exists a real number, named 1, such that $a \cdot 1 = a = 1 \cdot a$.
(9) If $a \neq 0$, then there exists a real number a^{-1} (the reciprocal of a) such that $a \cdot a^{-1} = 1 = a^{-1} \cdot a$. We sometimes write $a \cdot b^{-1}$ as a/b or $\frac{a}{b}$.
(10) The distributive property holds: $a(b + c) = ab + ac$.

Remark. These properties imply that $\mathbb{R}$ is a **field**.

A **complex number** is an ordered pair $z = (a, b)$ of real numbers (ordered: (a, b) does not necessarily equal (b, a)). We call a the **real part**, written as $\mathrm{Re}(z)$, and b the **imaginary part** of z, written as $\mathrm{Im}(z)$. Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ be complex numbers. We write

$$x = y \iff a_1 = a_2 \text{ and } b_1 = b_2.$$

We define addition and multiplication on complex numbers by

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2), \qquad z_1 z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1). \tag{1.1}$$

We observe that if $b = d = 0$, then

$$z_1 + z_2 = (a_1 + a_2, 0), \qquad z_1 z_2 = (a_1 a_2, 0),$$

so we can think of z_1 and z_2 as being just like real numbers when their imaginary parts equal zero.

Observe that $(a, b) = (a, 0) + (0, b)$ and $(0, 1)(b, 0) = (0, b)$. Thus we can write

$$z = (a, b) = (a, 0) + (0, 1)(b, 0).$$

It will be convenient to introduce the shorthand $i = (0, 1)$ and $(a, 0) = a$. Therefore, we can represent any complex number $z = (a, b)$ as

$$z = a + ib.$$

With these conventions, we will show that the properties of $\mathbb{R}$ outlined in Proposition 1.1 also holds for the set of complex numbers, which we denote by $\mathbb{C}$.

Proposition 1.2. *Let z_1, z_2, z_3 be complex numbers.*

(1) $z_1 + z_2$ *and* $z_1 z_2$ *are also complex numbers (closure).*
(2) *Addition is associative:* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
(3) *Addition is commutative:* $z_1 + z_2 = z_2 + z_1$.
(4) *There exists a complex number, named* $0 = (0, 0)$, *such that* $z_1 + 0 = z_1 = 0 + z_1$.
(5) *There exists a complex number* $-z_1$ *such that* $z_1 + (-z_1) = 0 = (-z_1) + z_1$. *For shorthand, we write* $z_1 + (-z_2)$ *as* $z_1 - z_2$.
(6) *Multiplication is associative:* $z_1(z_2 z_3) = (z_1 z_2)z_3$.
(7) *Multiplication is commutative:* $z_1 z_2 = z_2 z_1$.
(8) *There exists a complex number, named* $1 = (1, 0)$, *such that* $z_1 \cdot 1 = z_1 = 1 \cdot z_1$.
(9) *If* $z_1 \neq 0$, *then there exists a complex number* z_1^{-1} *(the reciprocal of* z_1*) such that* $z_1 \cdot z_1^{-1} = 1 = z_1^{-1} \cdot z_1$. *We sometimes write* $z_1 \cdot z_2^{-1}$ *as* z_1/z_2 *or* $\frac{z_1}{z_2}$.
(10) *The distributive property holds:* $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Proof. I'll prove (2) and (6). I leave the rest to you as an exercise. We will discuss (9) at length next class.

– (2): Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$. Using (1.1), we find that

$$z_1+z_2 = (a_1, b_1)+(a_2, b_2) = (a_1+a_2, b_1+b_2), \qquad z_2+z_1 = (a_2, b_2)+(a_1, b_1) = (a_2+a_1, b_2+b_1).$$

Since $a_1 + a_2 = a_2 + a_1$ and $b_1 + b_2 = b_2 + b_1$ by Proposition 1.1 (3), we have the desired equality.

– (6): Using (1.1), we first compute

$$\begin{aligned} z_1(z_2 z_3) &= (a_1, b_1)((a_2, b_2) \cdot (a_3, b_3)) \\ &= (a_1, b_1)(a_2 a_3 - b_2 b_3, a_2 b_3 + a_3 b_2) \\ &= (a_1(a_2 a_3 - b_2 b_3) - b_1(a_2 b_3 + a_3 b_2), a_1(a_2 b_3 + a_3 b_2) + b_1(a_2 a_3 - b_2 b_3)) \end{aligned}$$

A similar computation yields

$$(z_1 z_2)z_3 = ((a_1 a_2 - b_1 b_2)a_3 - (a_1 b_2 + a_2 b_1)b_3, (a_1 a_2 - b_1 b_2)b_3 + a_3(a_1 b_2 + a_2 b_1)).$$

Use Proposition 1.1 to check that these complex numbers are equal. □

Note that with these properties,

$$i^2 = (0, 1)^2 = (0, 1) = (-1, 0) = -1, \qquad i^3 = (i^2)i = -i, \qquad i^4 = i^3 \cdot i = -i \cdot i = -(-1) = 1.$$

2. January 8

Let us look at (9) from Proposition 1.2. First, given a complex number $z = \mathrm{Re}(z)+i\mathrm{Im}(z)$, we define the **conjugate** of z to be $\overline{z} = \mathrm{Re}(z) - i\mathrm{Im}(z)$. We compute

$$z\overline{z} = \overline{z}z = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2.$$

This leads us to the notion of the **modulus** of z, denoted $|z|$, which is $\sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$. We have the quick corollaries

$$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|, \qquad \mathrm{Im}(z) \leq |\mathrm{Im}(z)| \leq |z|.$$

Moreover, one can compute

$$z+\overline{z} = \mathrm{Re}(z)+i\mathrm{Im}(z)+(\mathrm{Re}(z)-i\mathrm{Im}(z)) = 2\mathrm{Re}(z), \quad z-\overline{z} = \mathrm{Re}(z)+i\mathrm{Im}(z)-(\mathrm{Re}(z)-i\mathrm{Im}(z)) = 2i\mathrm{Im}(z)$$

and

$$\overline{(\overline{z})} = \overline{\mathrm{Re}(z) - i\mathrm{Im}(z)} = \mathrm{Re}(z) + i\mathrm{Im}(z) = z$$

and

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{\mathrm{Re}(z_1) + i\mathrm{Im}(z_1) + \mathrm{Re}(z_2) + i\mathrm{Im}(z_2)} \\ &= \mathrm{Re}(z_1) + \mathrm{Re}(z_2) - i(\mathrm{Im}(z_1) + \mathrm{Im}(z_2)) = \overline{z_1} + \overline{z_2}\end{aligned}$$

and

$$|\overline{z}| = \sqrt{\mathrm{Re}(z)^2 + (-\mathrm{Im}(z))^2} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = |z|.$$

Lemma 2.1. *If z is a complex number, then $|z| \geq 0$, with equality if and only if $z = 0$.*

Proof. The first part is immediate from the definition of $|z|$. Now, suppose that $z = a + ib$ satisfies $|z| = 0$. This is equivalent to saying that $\sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = 0$, which is equivalent to saying that $\mathrm{Re}(z)^2+\mathrm{Im}(z)^2 = 0$. Since $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real, we have $\mathrm{Re}(z)^2 \geq 0$ (resp. $\mathrm{Im}(z)^2 \geq 0$), with equality if and only if $\mathrm{Re}(z) = 0$ (resp. $\mathrm{Im}(z) = 0$). Thus $\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ is equivalent to saying that $\mathrm{Re}(z) = \mathrm{Im}(z) = 0$, so $z = 0$. ☐

Suppose that $z \neq 0$. Then $|z| \neq 0$, and we can consider the product of the real number $1/|z|^2$ with the complex number $\overline{z}$:

$$\frac{1}{|z|^2} \cdot \overline{z} = \frac{1}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}(\mathrm{Re}(z) - \mathrm{Im}(z)i) = \frac{\mathrm{Re}(z)}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} + i\frac{-\mathrm{Im}(z)}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

But since $z \cdot (\frac{1}{|z|^2} \cdot \overline{z}) = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$. Thus we have proved that if $z = a + bi$, then

$$z^{-1} = \frac{\mathrm{Re}(z)}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} + i\frac{-\mathrm{Im}(z)}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

(The book gives a slightly different presentation which is ultimately equivalent.) Now, using our law for how to multiply complex numbers, we find that if $z_1 = a_1+b_1i$ and $z_2 = a_2+b_2i \neq 0$, then

$$\frac{z_1}{z_2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_2 - a_1b_2}{a_2^2 + b_2^2}.$$

Example 2.2.

$$\frac{2 - 3i}{4 + i} = \frac{(2 - 3i)(4 - i)}{(4 + i)(4 - i)} = \frac{5 - 14i}{17} = \frac{5}{17} + i\frac{-14}{17}.$$

Example 2.3. There is an analogue of the binomial formula: For each integer $n \geq 1$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad 0! = 1.$$

(The proof is an exercise.)

There is an important geometric interpretation of complex numbers. In particular, we can associate $z = (a, b)$ with the vector in $\mathbb{R}^2$ starting at $(0, 0)$ and ending at (a, b). With this interpretation, our rule of addition

$$z_1 + z_2 = (a_1 + b_1, a_2 + b_2)$$

corresponds with the usual notion of vector addition. (Draw the parallelogram picture.) Also, the modulus $|z|$ corresponds with the length of the vector. Moreover, if $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$, then the distance between the points (a_1, b_1) and (a_2, b_2) in $\mathbb{R}^2$ is

$$\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} = |z_1 - z_2|.$$

Negating a complex number $-z = (-a, -b)$ can be seen geometrically as reflecting the vector corresponding with z across the origin.

Recall the equation for a circle with center (x_0, y_0) and radius $r > 0$: $(x - x_0)^2 + (y - y_0)^2 = R^2$. But thinking of $z = x + iy$ and $z_0 = x_0 + iy_0$, we observe that

$$R^2 = (x - x_0)^2 + (y - y_0)^2 = |z - z_0|^2.$$

Thus the equation for the circle becomes $|z - z_0| = R$.

Example 2.4. The equation $|z - 3 + 2i| = 1$ represents the circle centered at $z_0 = (3, -2)$ with radius 1.

Lemma 2.5 (Triangle inequality)**.** *If z_1, z_2 are complex numbers, then $|z_1 + z_2| \leq |z_1| + |z_2|$.*

Proof. We compute

$$\begin{aligned} |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + z_2\overline{z_1}) \\ &= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2}) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |\overline{z_2}| \\ &= |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| = (|z_1| + |z_2|)^2. \end{aligned}$$

Since the modulus of a complex number is nonnegative, we can take the square root of both sides, and the conclusion follows. □

Using induction, one can prove for any $n \geq 1$ that

$$|z_1 + \cdots + z_n| \leq |z_1|^2 + \cdots + |z_n|^2.$$

Lemma 2.6. *If z_1, z_2 are complex numbers, then $|z_1 - z_2| \geq ||z_1| - |z_2||$.*

Proof. Since $|z_1| - |z_2|$ is a real number, we compute

$$\begin{aligned} ||z_1| - |z_2||^2 = (|z_1| - |z_2|)^2 &= |z_1|^2 + |z_2|^2 - 2|z_1| \cdot |z_2| \\ &\leq |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1} = (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = |z_1 - z_2|^2. \end{aligned}$$

□

3. January 10

Let (r, θ) be the polar coordinates of the point (x, y) corresponding to a *nonzero* complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, we can write z in **polar form**

$$z = r(\cos\theta + i\sin\theta).$$

Recall that if (x, y) corresponds with $z = x + iy$, then $|z|$ is the length of the vector from $(0, 0)$ to (x, y). So in fact $r = |z|$. The real number θ represents the angle (radians) that z makes with the positive real axis when interpreting z as a vector. This is determined up to an integer multiple of 2π by means of the equation

$$\tan\theta = y/x,$$

where the quadrant containing z must be specified. Each value of θ satisfying these equations is called an **argument** of z, and the set of all such values is called $\arg(z)$. The **principal value** of $\arg z$, denoted $\mathrm{Arg}\ z$, is the unique value Θ such that $-\pi < \Theta \leq \pi$. Thus

$$\arg z = \{\mathrm{Arg}\ z + 2n\pi \colon n \in \mathbb{Z}\}.$$

Example 3.1. The complex number $z = -1 - 2i$ lies in the third quadrant. Thus $\mathrm{Arg}\ z = -2\pi/3$ and $\arg z = \{-2\pi/3 + 2\pi n \colon n \in \mathbb{Z}\}$.

Definition 3.2. *We define the symbol $e^{i\theta}$ to equal* $\cos\theta + i\sin\theta$.

This definition requires some discussion, which will be made rigorous later on. Recall the everywhere absolutely convergent Taylor series expansions

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}, \qquad \cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}.$$

Now, since $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$, we find that

$$\begin{aligned}\cos\theta + i\sin\theta &= \Big(1 - \frac{\theta^2}{2} + \cdots + \frac{(-1)^n\theta^{2n}}{(2n!)}\Big) + i\Big(\theta - \frac{\theta^3}{6} + \cdots + \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\Big)\\ &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \cdots + \frac{(i\theta)^n}{n!} + \cdots,\end{aligned}$$

which looks like the everywhere absolutely convergent Taylor expansion for e^x evaluated at $x = i\theta$. Of course, we have not developed Taylor series for complex variables yet.

This leads to a more compact expression of the polar form of z, the **exponential form**:

$$z = re^{i\theta}.$$

Example 3.3. Let $z = -1 - 2i$. Then $|z| = \sqrt{5}$ and $\mathrm{Arg}\ z = -2\pi/3$. Hence $z = \sqrt{5}e^{-2\pi i/3}$. Of course, for each $n \in \mathbb{Z}$, we also have $z = \sqrt{5}e^{-2\pi i/3 + 2\pi i n}$.

Using standard trigonometric identities, we find that

$$\begin{aligned}e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}.\end{aligned}$$

Hence if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}, \qquad z_1/z_2 = r_1/r_2 e^{i(\theta_1-\theta_2)}.$$

In particular, if $z = re^{i\theta}$, then
$$z^{-1} = 1/z = r^{-1}e^{-i\theta}.$$

Lemma 3.4. *If $z = re^{i\theta}$, then for each integer n, we have $z^n = r^n e^{in\theta}$, with the convention that $z^0 = 1$.*

Proof. First, we prove that if $n \geq 1$, then $z^n = r^n e^{in\theta}$. For $n = 1$, we recover the polar form already mentioned above. So suppose that $z^n = r^n e^{in\theta}$ for some integer $n \geq 1$. This inductive hypothesis implies that
$$z^{n+1} = z^n z = (r^n e^{in\theta})(re^{i\theta}) = r^{n+1}e^{in\theta}e^{i\theta} = r^{n+1}e^{i(n+1)\theta}.$$
Hence, by mathematical induction, we have the claimed result for $n \geq 1$. A similar application of mathematical induction will indicate that $z^{-n} = r^{-n}e^{-in\theta}$ for all integers $n \geq 1$, which completes the proof. □

Example 3.5. Let $z = -1-2i$. Then $z = \sqrt{5}e^{-2\pi i/3}$, and $z^6 = \sqrt{5}^6 e^{(-2\pi i/3)\cdot 6} = 5^3 e^{-4\pi i} = 5^3$.

Note that when $r = 1$, the above lemma indicates that $(e^{i\theta})^n = e^{in\theta}$ for each integer n. Hence we arrive at **de Moivre's formula**
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example 3.6. de Moivre's formula with $n = 2$ implies that $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$. Hence $\cos^2\theta - \sin^2\theta + i(2\sin\theta\cos\theta) = \cos(2\theta) + i\sin(2\theta)$. Equating the real and imaginary parts, we arrive at the familiar identities
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

4. January 13

For $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the expression $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ leads to the important identity for arguments:

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \qquad \arg(z_1/z_2) = \arg(z_1) - \arg(z_2).$$

However, this identity does not hold for Arg; an easy counterexample can be seen with $\mathrm{Arg}(-1) = \pi$ and $\mathrm{Arg}(i) = \pi/2$.

We observe [DRAW A PICTURE] that if $z_1 = r_1 e^{i\theta}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 - \theta_2 = 2k\pi$ for some integer k. In other words,

$$z_1 = z_2 \iff |z_1| = |z_2| \qquad \text{and} \qquad \arg z_1 = \arg z_2.$$

We use this to solve the equation $z^n = z_0$, where z_0 is a given complex number and $n \neq 0$ is an integer. We write $z_0 = r_0 e^{i\theta_0}$, and we have $z^n = r^n e^{in\theta}$. Hence we must have $r^n = r_0$, so $r = r_0^{1/n}$, and there must exist an integer k such that $n\theta - \theta_0 = 2\pi k$. Solving for θ, we must have

$$\theta = \frac{\theta_0}{n} + \frac{2\pi k}{n}.$$

Hence our solutions to the equation $z^n = z_0 = r_0 e^{i\theta_0}$ are

$$z = r_0^{1/n} \exp\Big[i\Big(\frac{\theta_0}{n} + \frac{2k\pi}{n}\Big)\Big], \qquad k \in \mathbb{Z}.$$

All of these solutions lie on the circle $|z| = r_0^{1/n}$, and they are equally spaced every $2\pi/n$ radians, starting at θ_0/n. Thus all of the *distinct* roots are obtained by considering $k = 0, 1, \dots, n-1$, and for other values of k, we obtain repeats of these distinct roots. Hence the set of distinct solutions to the equation $z^n = z_0$ are

$$c_k = r_0^{1/n} \exp\Big[i\Big(\frac{\theta_0}{n} + \frac{2k\pi}{n}\Big)\Big], \qquad k = 0, 1, \dots, n-1.$$

Example 4.1. Let us find the three cube roots of $-27i$. We first write

$$-27i = 27 \exp\Big[i\Big(-\frac{\pi}{2} + 2\pi k\Big)\Big], \qquad k \in \mathbb{Z}$$

and conclude that the desired roots are

$$c_k = 3 \exp\Big[i\Big(-\frac{\pi}{6} + \frac{2k\pi}{3}\Big)\Big], \qquad k = 0, 1, 2.$$

In rectangular form, these are $c_0 = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$, $c_1 = 3i$, $c_2 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$.

Example 4.2. Let $n \geq 1$ be an integer, and let us find the n-th roots of 1. These are often called the **n-th roots of unity**. Thus we want to find the solutions to $z^n = 1$, in which case $r_0 = 1$ and we may take $\theta_0 = 0$. Then the solutions are given by

$$1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}, \qquad \omega = \exp(2\pi i/n).$$

Example 4.3. Using de Moivre's formula, we know that

$$\cos(5\theta) + i\sin(5\theta) = (\cos\theta + i\sin\theta)^5,$$

which, upon binomial expansion, equals

$$\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta).$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, we equate the real and imaginary parts to find

$$\cos(5\theta) = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta, \qquad \sin(5\theta) = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

Choosing $\theta = \pi/5$ for the sine identity and substituting $x = \sin(\pi/5)$ and $y = x^2$, we arrive at the equation $0 = x(5 - 20x^2 + 16x^4) = x(5 - 20y + 16y^2)$. Therefore, we can solve for x, hence $\sin(\pi/5)$, using the quadratic formula and the fact that $\sin(\pi/5) < \sin(\pi/4) = 1/\sqrt{2}$. We conclude that

$$\sin(\pi/5) = \sqrt{\frac{5-\sqrt{5}}{8}},$$

and thus by $\sin^2(\pi/5) + \cos^2(\pi/5) = 1$ we conclude that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}.$$

Hence we can find the rectangular coordinates for the fifth roots of unity (I leave to you).

We begin by covering some vocabulary on region in the plane that we will use regularly throughout the course.

Definition 4.4. *An* ε**-neighborhood (centered at** z_0**)** *is a set of the shape* $\{z : |z - z_0| < \varepsilon\}$. *(One often drops the* ε *and simply uses "neighborhood". A* **deleted neighborhood** *is a set of the shape* $\{z : 0 < |z - z_0| < \varepsilon\}$.

In what follows, let S be a region in the complex plane; this can also be viewed as a set of complex numbers.

5. January 15

Definition 5.1. *A point z_0 is an* **interior point** *of S if there exists a neighborhood centered at z_0 which contains only points in S. The point z_0 is an* **exterior point** *of S if there exists a neighborhood centered at z_0 which consists only of points not in S. A point z_0 which is neither an interior point nor an exterior point is called a* **boundary point***; every neighborhood of a boundary point contains at least one point in S and at least one point not in S. The union of all boundary points is called the* **boundary** *of S.*

Example 5.2. Let $S = \{z\colon |z| < 1\}$. The point $z = 1/2$ is an interior point of S because S contains the neighborhood $\{z\colon |z - \frac{1}{2}| < \frac{1}{2}\}$. The point $z = 3/2$ is an exterior point of S because the neighborhood $\{z\colon |z - \frac{3}{2}| < \frac{1}{2}\}$ contains no point within S. The point $z = 1$ is a boundary point. To see this, fix $\varepsilon > 0$. Note that the set $\{z\colon |z - 1| < \varepsilon\}$ contains $1 - \frac{\varepsilon}{2}$ (which lies in S) and $1 + \frac{\varepsilon}{2}$ (which does not lie in S). Since ε was arbitrary, each neighborhood of $z = 1$ contains at least one point in S and at least one point not in S.

Definition 5.3. *The set S is* **open** *if it does not contain any of its boundary points. The set S is* **closed** *if it contains all of its boundary points. The* **closure** *of a set S is the closed set containing all points in S and all of the boundary points of S.*

Example 5.4. Let $S = \{z\colon |z| < 1\}$. This set is open: If $z_0 \in S$, then $\{z\colon |z-z_0| < 1-|z_0|\}$ is a neighborhood of z_0 containing only points in S. Note that if $|z_0| > 1$, then the neighborhood $\{z\colon |z - z_0| < |z_0| - 1\}$ contains only points outside of S. Hence each such z_0 is an exterior point. We conclude that the boundary of S is $\{z\colon |z| = 1\}$.

Remark. Sets of points in the complex plane can be neither closed nor open, like the punctured disk $\{z\colon 0 < |z| \leq 1\}$.

Definition 5.5. *An open set S is* **(path)-connected** *if any two points can be connected by a path without exiting S.*

Example 5.6. The sets $\{z\colon |z| < 1\}$ and $\{z\colon 1 < |z| < 2\}$ are each both open and connected.

Definition 5.7. *A set S is* **bounded** *if there exists a finite real number $R > 0$ such that every point z in S satisfies $|z| < R$.*

Example 5.8. The sets $\{z\colon |z| < 1\}$ and $\{z\colon 1 < |z| < 2\}$ are bounded, but the set $\{z\colon \mathrm{Im}(z) > 0\}$ is not bounded.

Definition 5.9. *An* **accumulation point***, or* **limit point***, of a set S is a point z_0 such that each deleted neighborhood of z_0 contains at least one point in S.*

I leave it as an exercise to prove that S is closed if and only if it contains all of its accumulation points.

Quiz

6. January 17

With this vocabulary in place, we can now start to develop the theory of functions of a complex variable. Let S be a set of complex numbers. A **function** f defined on S is a rule that assigns to each z in S a complex number w. We will often use the shorthand $f : S \to \mathbb{C}$. The set S is called the **domain** of f. When the rule f is specified but S is not, then we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes a given function from its values.

Example 6.1. Let $S = \mathbb{C} - \{0\}$. Consider the function $f : S \to \mathbb{C}$ given by $f(z) = 1/z$. We might simply refer to the function as $1/z$ with largest domain (which is $\mathbb{C} - \{0\}$).

Suppose that $f(x + iy) = u + iv$, where x, y, u, v are real numbers. Then each of the real numbers u and v depend on the real numbers x and y. We sometimes phrase this as

$$f(z) = u(x, y) + iv(x, y).$$

If the function $v(x, y)$ always equals zero, then $f(x + iy)$ is always real, in which case f is a **real-valued function** of a complex variable. We can play the same game in polar coordinates: Write $z = re^{i\theta}$ instead of $z = x + iy$, in which case $f(re^{i\theta}) = u + iv$ and

$$f(z) = u(r, \theta) + iv(r, \theta).$$

Example 6.2. If $f(z) = z^2$, then in writing $z = x + iy$, we find that $f(x + iy) = u(x, y) + iv(x, y)$, where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We can also think about this in polar coordinates. Writing z as $re^{i\theta}$, we find

$$f(z) = z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta} = r^2(\cos(2\theta) + i \sin(2\theta)).$$

Thus $u(r, \theta) = r^2 \cos(2\theta)$ and $v(r, \theta) = r^2 \sin(2\theta)$.

Example 6.3. The function $f(z) = |z|^2$ is an example of a real-valued function.

If $n \geq 1$ is a positive integer and $a_0, a_1, \dots, a_n$ are complex numbers, then we call

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

a **polynomial of degree** n. The quotient $P(z)/Q(z)$ of two polynomials $P(z)$ and $Q(z)$ is called a **rational function**, and it is defined whenever $Q(z) \neq 0$.

We can generalize the concept of a function to incorporate a rule that assigns more than one value to a point z in the domain of definition. These so-called **multiple-valued functions** occur frequently in complex analysis.

Example 6.4. Let $z = re^{i\theta}$ with principal argument Θ. From our earlier work, we see that $z^{1/2}$ has two values, namely

$$z^{1/2} = \sqrt{|z|} \exp[i(\Theta/2 + \pi k)], \qquad k = 0, 1.$$

Since $e^{\pi i} = -1$, we in fact have

$$z^{1/2} = \pm\sqrt{|z|} \exp(i\Theta/2).$$

Thus $z^{1/2}$ is multiple-valued. But, if we only choose the positive value of $\pm\sqrt{|z|}$ and write

$$(*)\ f(z) = \sqrt{|z|} \exp(i\Theta/2),$$

then the (single-valued) function $(*)$ is well-defined on the nonzero numbers in the complex plane. Since zero is the only square root of zero, we also write $f(0) = 0$, so that the function $(*)$ is well-defined on all of $\mathbb{C}$.

For a function f whose domain is a subset $U \subset \mathbb{C}$ that sends numbers in U to numbers in a subset $V \subset \mathbb{C}$, we have the shorthand $f : U \to V$. We sometimes say that f is a **map** or **mapping** from U to V. If the pair $(u, v) \in U \times V$ satisfies $v = f(u)$, then we say that v is the **image** of u under the mapping f. Sometimes, if we know f explicitly, I might write $z \mapsto f(z)$ to denote the function (like $z \mapsto \sqrt{z}$).

Definition 6.5. *If $f : U \to V$ has the property that for any pair of numbers $u_1 \neq u_2$ in U we have $f(u_1) \neq f(u_2)$, then the mapping f is* **injective** *(or* **one-to-one***). If for every $v \in V$ there exists a $u \in U$ such that $v = f(u)$, then the mapping f is* **surjective** *(or* **onto***). If a mapping is both injective and surjective, then the mapping is called* **bijective**.

If $f : U \to V$ is not injective, then a given $v \in V$ could be the image of many $u \in U$. Hence we define the **inverse image** (or **preimage**) of $v \in V$ to be $\{u \in U : f(u) = v\}$.

Assignment: Look at Section 14, study the mapping $z \mapsto z^2$.

Notation 6.6. Let U, V be sets. The notation $u \in U$ means that an element u lies in U. The notation $U \subset V$ or $U \subseteq V$ means that U is a subset of V (that is, if $u \in U$, then $u \in V$).

Definition 6.7. *Let $z_0 \in \mathbb{C}$, and let f be a function defined on some deleted NBHD of z_0. The statement "$f(z)$ has* **limit** *w_0 as z approaches z_0", written as*

$$\lim_{z \to z_0} f(z) = w_0,$$

if for all $\varepsilon > 0$, we can find some $\delta > 0$ (maybe depending on ε) such that

$$|f(z) - w_0| < \varepsilon \qquad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

(Note that we can always make δ smaller if necessary.)

DRAW A PICTURE, TALK ABOUT THIS FOR A LITTLE WHILE

Example 6.8. Let $f(z) = z + \overline{z}$. I claim that

$$\lim_{z \to i} f(z) = 0.$$

Let $\varepsilon > 0$. We first compute

$$|f(z) - 0| = |z + \overline{z}| = |z - i + \overline{z} + i| = |(z - i) + \overline{z - i}| \leq 2|z - i|.$$

Let $\delta = \varepsilon/2$. If $0 < |z - i| < \delta$, then $|f(z) - 0| < \varepsilon$.

7. January 20

MLK Jr Day

8. January 22

Theorem 8.1. *When the limit of a function $f(z)$ exists at a point z_0, the limit is unique.*

Proof. Suppose that

$$\lim_{z\to z_0} f(z) = w_0 \qquad \text{and} \qquad \lim_{z\to z_0} f(z) = w_1.$$

Thus for all $\varepsilon > 0$, there exist numbers $\delta_0, \delta_1 > 0$ such that

$$|f(z) - w_0| < \varepsilon/2 \qquad \text{whenever} \quad 0 < |z - z_0| < \delta_0.$$

and

$$|f(z) - w_1| < \varepsilon/2 \qquad \text{whenever} \quad 0 < |z - z_0| < \delta_1.$$

Now, if $|z - z_0| < \min\{\delta_0, \delta_1\}$, then

$$|w_0 - w_1| = |w_0 - f(z) + f(z) - w_1| \leq |w_0 - f(z)| + |w_1 - f(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, we have $w_0 = w_1$. □

Example 8.2. Let $f(z) = z/\overline{z}$. I claim that

$$\lim_{z\to 0} f(z)$$

does not exist. Note that if $x \neq 0$ is real, then $f(x) = 1$, and if $y \neq 0$ is real, then $f(iy) = -1$. Thus as we approach $z = 0$ from two different directions, the limits are different. Since the limit as $z \to 0$ is unique if it exists, we must conclude that $\lim_{z\to 0} f(z)$ does not exist.

Theorem 8.3. *Let $z = x + iy$, $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, and $f(z) = u(x, y) + iv(x, y)$. We have that*

$$\lim_{z\to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Proof. See Section 16. The result is useful, but the proof is tedious. □

Theorem 8.4. *Suppose that*

$$\lim_{z\to z_0} f(z) = w_0, \qquad \lim_{z\to z_0} F(z) = W_0.$$

Then

$$\lim_{z\to z_0} (f(z) + F(z)) = w_0 + W_0, \qquad \lim_{z\to z_0} f(z)F(z) = w_0 W_0,$$

and, if $W_0 \neq 0$,

$$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

Proof. Use the previous theorem to reduce the proof to statements about limits of functions in two real variables, then use properties of limits from multivariable calculus. □

Example 8.5. By induction, one can prove that if $n \geq 1$ is an integer and $z_0 \in \mathbb{C}$, then $\lim_{z\to z_0} z^n = z_0^n$. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial, then for any $z_0 \in \mathbb{C}$, $\lim_{z\to z_0} p(z) = p(z_0)$.

Quiz...

9. January 24

It is sometimes convenient to include the **point at infinity**, denoted ∞, with the complex plane (and to use limits involving it).

STEREOGRAPHIC PROJECTION PICTURE, ∞ IS THE NORTH POLE N, UNIT CIRCLE CENTERED AT $z = 0$, **RIEMANN SPHERE**

Definition 9.1. *Let $\varepsilon > 0$. We call $\{z \in \mathbb{C}\colon |z| > 1/\varepsilon\}$ a* ***neighborhood of*** ∞*.*

CONVENTION: If we refer to a point z in the complex plane, then we refer to a point that is not ∞. If ∞ is considered, it will be explicitly mentioned.

Theorem 9.2. *If $z_0, w_0 \in \mathbb{C}$, then:*

(1) $\lim_{z\to z_0} f(z) = \infty$ *if* $\lim_{z\to z_0} 1/f(z) = 0$.
(2) $\lim_{z\to\infty} f(z) = w_0$ *if* $\lim_{z\to 0} f(1/z) = w_0$.
(3) $\lim_{z\to\infty} f(z) = \infty$ *if* $\lim_{z\to 0} 1/f(1/z) = 0$.

Proof. I'll prove the first part; the others are similar. If $\lim_{z\to z_0} 1/f(z) = 0$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|1/f(z) - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta.$$

We rewrite this as

$$|f(z)| > 1/\varepsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta.$$

This a restatement of $\lim_{z\to z_0} f(z) = \infty$. □

Example 9.3. We have

$$\lim_{z\to\infty} \frac{z+2}{z-1} = 1 \quad \text{since} \quad \lim_{z\to 0} \frac{(1/z)+2}{(1/z)-1} = \lim_{z\to 0} \frac{1+2z}{1-z} = 1.$$

Definition 9.4. *A function f is continuous at a point z_0 if $\lim_{z\to z_0} f(z)$ exists, $f(z_0)$ exists, and $\lim_{z\to z_0} f(z) = f(z_0)$. In the $\delta - \varepsilon$ language, a function f is continuous at a point z_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.*

From our limit arithmetic discussed last class, we have that the sum, difference, product, and quotient of two functions which are continuous at z_0 is also continuous at z_0. The following is much more subtle, which I will not prove.

Theorem 9.5. *If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

As with limits, we can establish continuity by looking at the real and imaginary parts of f.

Theorem 9.6. *Write $f(z) = f(x+iy) = u(x,y)+iv(x,y)$. The functions u, v are continuous at (x_0, y_0) if and only if f is continuous at $z_0 = x_0 + iy_0$.*

Definition 9.7. *If a function $f : U \to \mathbb{C}$ is continuous at each point $z \in U$, then f is continuous on U.*

One of the key results for continuous functions of a real variable is that if $f : [a, b] \to \mathbb{R}$ is continuous, then f achieves its maximum and minimum. A related result for complex-valued functions is:

Theorem 9.8. *Let $U \subseteq \mathbb{C}$ be closed and bounded, and let f be continuous on U. There exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in U$, where equality holds for at least one such z. We then say that f is* **bounded on** U.

This follows from applying an analogous result for functions from $\mathbb{R}^2$ to $\mathbb{R}$ to $u(x,y)$ and $v(x,y)$.

Definition 9.9. *Let f be a function whose domain contains a neighborhood $\{z \in \mathbb{C} \colon |z-z_0| < \varepsilon\}$ of a point z_0. The* **derivative** *of f at z_0 is the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

The function f is **differentiable** *at z_0 if $f'(z_0)$ exists. If f is differentiable at all points in a region $U \subseteq \mathbb{C}$, then f is differentiable on U.*

Example 9.10. Let $f(z) = 1/z$ and $z_0 \neq 0$. If $z \neq z_0$, then

$$\frac{z^{-1} - z_0^{-1}}{z - z_0} = -\frac{1}{z_0 z}.$$

By the properties of limits, the limit as $z \to z_0$ is $-1/z_0^2$. Then f is differentiable on $\mathbb{C} - \{0\}$.

Example 9.11. Let $f(z) = |z|^2$ and $z_0 \in \mathbb{C}$. If $z \neq z_0$ and $z_0 = 0$, then

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{|z|^2}{z} = \overline{z},$$

whose limit is zero as $z \to 0$. On the other hand if $z_0 \neq 0$, then

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z\overline{z} - z_0\overline{z_0}}{z - z_0} = \frac{z\overline{z} - z\overline{z_0} + z\overline{z_0} - z_0\overline{z_0}}{z - z_0} = z\frac{\overline{z - z_0}}{z - z_0} + \overline{z}_0$$

If z traverses the complex numbers such that $\mathrm{Im}(z) = \mathrm{Im}(z_0)$, then the limit is

$$\lim_{z \to z_0} \left(z\frac{\overline{z - z_0}}{z - z_0} + \overline{z}_0 \right) = z_0 + \overline{z}_0 = 2\mathrm{Re}(z_0).$$

If z traverses the complex numbers such that $\mathrm{Re}(z) = \mathrm{Re}(z_0)$, then

$$\lim_{z \to z_0} \left(z\frac{\overline{z - z_0}}{z - z_0} + \overline{z}_0 \right) = -z_0 + \overline{z}_0 = 2\mathrm{Re}(z_0) = -2i\mathrm{Im}(z_0).$$

Since $z \neq z_0$, it follows from the uniqueness of limits that $\lim_{z \to z_0} |z|^2$ does not exist.

10. January 27

Theorem 10.1. *If $f'(z_0)$ exists, then f is continuous at z_0.*

Proof. We compute

$$\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0) = f'(z_0)\cdot 0 = 0.$$

□

All of the differentiation rules from calculus carry forward to functions of a complex variable because the limit definition is the same in both settings:

$$(fg)' = fg' + f'g, \qquad (f/g)' = (gf' - fg')/g^2, \qquad (f\circ g)' = (f'\circ g)\cdot g'.$$

Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, and suppose that $f'(z_0)$ exists. We will begin the setup for a critical relationship between the derivative of f (with respect to z) and the derivatives of u and v (with respect to x and y). To do this, we recast the derivative as

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}, \qquad \Delta z\in\mathbb{C}$$

and introduce some notation:

$$z_0 = x_0 + iy_0, \qquad \Delta z = \Delta x + i\Delta y,$$

in which case

$$f(z_0+\Delta z) - f(z_0) = [u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y)] - [u(x_0,y_0) + iv(x_0,y_0)]$$

Now,

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left(\frac{u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)}{\Delta x+i\Delta y} + i\frac{v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)}{\Delta x+i\Delta y}\right)\end{aligned}$$

Since $f'(z_0)$ exists, the above quotient will tend to the same value regardless of the manner in which $\Delta x\to 0$ and $\Delta y\to 0$. First, suppose that $\Delta y = 0$, in which case it remains for $\Delta x\to 0$. Upon substituting 0 for Δy, we arrive at

$$f'(z_0) = \lim_{\Delta x\to 0}\left(\frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x} + i\frac{v(x_0+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}\right) = u_x(x_0,y_0) + iv_x(x_0,y_0).$$

Now, suppose that $\Delta x = 0$, in which case it remains for $\Delta y\to 0$. Upon substituting 0 for Δx, we arrive at

$$f'(z_0) = \lim_{\Delta y\to 0}\left(\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{i\Delta y} + i\frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{i\Delta y}\right) = -iu_y(x_0,y_0) + v_y(x_0,y_0).$$

Equating the real and imaginary parts yields the **Cauchy–Riemann equations**

$$u_x(x_0,y_0) = v_y(x_0,y_0) \qquad \text{and} \qquad u_y(x_0,y_0) = -v_x(x_0,y_0).$$

Theorem 10.2. *If $f(z) = u(x,y) + iv(x,y)$ and f' exists at $z_0 = x_0 + iy_0$, then u and v must satisfy the Cauchy–Riemann equations*

$$u_x = v_y, \qquad u_y = -v_x.$$

Moreover, $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$.

Example 10.3. Let $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) = u + iv$. We already know that f is differentiable on $\mathbb{C}$. We compute $u_x = 2x$, $u_y = -2y$, $v_x = 2x$, $v_y = 2y$. Then $u_x = v_y$ and $u_y = -v_x$, checking that f satisfies the Cauchy–Riemann equations. Moreover, $2z = 2(x+iy) = 2x + i(2y) = u_x + iv_x = f'(z)$.

Example 10.4. Let $f(z) = |z|^2 = x^2 + y^2$. Then $u_x = 2x$, $u_y = 2y$, $v_x = v_y = 0$. The Cauchy–Riemann equations are only satisfied at the point $x_0 = y_0 = 0$, which, as we showed yesterday, is the only point at which f is differentiable.

Right now, the Cauchy–Riemann equations are only suitable for showing where f **is not** differentiable. We will see that they can actually show us where f **is** differentiable.

Theorem 10.5. *Let $f(z) = u(x,y) + iv(x,y)$ be defined in some ε NBHD of a point $z_0 = x_0 + iy_0$. Suppose that*

(1) u_x, u_y, v_x, v_y exist and are continuous everywhere in the NBHD, and
(2) these partial derivatives satisfy the Cauchy–Riemann equations at (x_0, y_0).

Then $f'(z_0)$ exists and equals $u_x(x_0,y_0) + iv_x(x_0,y_0)$.

We begin with some notation that will help clean up the proof. Write

$$\Delta x = x - x_0, \qquad \Delta y = y - y_0, \qquad \Delta z = z - z_0 = \Delta x + i\Delta y.$$

If $F : \mathbb{R}^2 \to \mathbb{R}$ is a function differentiable at (x_0, y_0), then this is equivalent in a NBHD of (x_0, y_0) to having the Taylor expansion

$$F(x,y) = F(x_0,y_0) + F_x(x_0,y_0)\Delta x + F_y(x_0,y_0)\Delta y + E_1(\Delta x)\Delta x + E_1(\Delta y)\Delta y,$$

where $E_1 \to 0$ (resp. $E_2 \to 0$) as $\Delta x \to 0$ (resp. $\Delta y \to 0$).

Proof. Let $\varepsilon > 0$. Assume conditions (1) and (2) for u and v. We then have

$$f(z) - f(z_0) = [u(x,y) + iv(x,y)] - [u(x_0,y_0) + iv(x_0,y_0)].$$

By assuming condition (2) for u and v, we have that

$$\begin{aligned} u(x,y) - u(x_0,y_0) &= u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + E_1(\Delta x)\Delta x + E_2(\Delta y)\Delta y, \\ v(x,y) - v(x_0,y_0) &= v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + E_3(\Delta x)\Delta x + E_4(\Delta y)\Delta y. \end{aligned}$$

Thus

$$\begin{aligned} f(z) - f(z_0) = {}& u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + i[v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y] \\ &+ E_5(\Delta x)\Delta x + E_6(\Delta x)\Delta x. \end{aligned}$$

By assuming condition (1) for u and v (so $u_x = v_y$ and $u_y = -v_x$), we can rewrite this as

$$\begin{aligned} f(z) - f(z_0) &= u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + i[v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y] \\ &\quad + E_5(\Delta x)\Delta x + E_6(\Delta x)\Delta x \\ &= u_x(x_0,y_0)(\Delta x + i\Delta y) + iv_x(x_0,y_0)(\Delta x + i\Delta y) \\ &\quad + E_5(\Delta x)\Delta x + E_6(\Delta x)\Delta x \\ &= [u_x(x_0,y_0) + iv_x(x_0,y_0)]\Delta z + E_5(\Delta x)\Delta x + E_6(\Delta x)\Delta x. \end{aligned}$$

Dividing through by $z - z_0 = \Delta z$, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0,y_0) + iv_x(x_0,y_0) + E_5(\Delta x)\Delta x/\Delta z + E_6(\Delta y)\Delta y/\Delta z.$$

Now, observe that since $\Delta z = \Delta x + i\Delta y$, we have $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$ by the triangle inequality. Moreover, $|E_5(\Delta x) + E_6(\Delta y)| \to 0$ as $\Delta z \to 0$, i.e., as $z \to z_0$. This was the desired conclusion. $\square$

11. January 29

Example 11.1. Let $f(z) = f(x+iy) = e^x \cos y + ie^x \sin y$. Then $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$, and we compute $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = e^x \sin y$, $v_y = e^x \cos y$. We see that u, v are continuously differentiable on all of $\mathbb{R}^2$, and $u_x = v_y$ and $u_y = -v_x$ on all of $\mathbb{R}^2$. Thus $f(z)$ is differentiable everywhere, and $f'(z) = f(z)$. Notice that $f(z) = e^x e^{iy}$.

Example 11.2. Let $f(z) = f(x+iy) = x^3 + i(1-y)^3$. Then u, v are continuously differentiable on all of $\mathbb{R}^2$, and $u_x = 3x^2$, $u_y = 0$, $v_x = 0$, $v_y = -3(1-y)^2$. We have $u_y = -v_x$ trivially, and $u_x = v_y$ if and only $x^2 + (1-y)^2 = 0$. This is only satisfied at the point $(x_0, y_0) = (0, 1)$, corresponding with $z_0 = i$. The derivative here is $f'(0 + i \cdot 1) = 3 \cdot 0^2 + 0 = 0$.

There is a polar version of the Cauchy–Riemann equations. Write $0 \neq z = x + iy = re^{i\theta}$. Since $e^{i\theta} = \cos\theta + i \sin\theta$, we equate real and imaginary parts to obtain $x = r\cos\theta$ and $y = r\sin\theta$. We also write

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta).$$

Assuming that u, v are continuously differentiable at some point $z_0 = x_0 + iy_0 = r_0 e^{i\theta_0}$, we compute via the multivariable chain rule that

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta, \qquad u_\theta = u_x x_\theta + u_y y_\theta = -u_x r \sin\theta + u_y r\cos\theta \\ v_r &= v_x x_r + v_y y_r = v_x \cos\theta + v_y \sin\theta, \qquad v_\theta = v_x x_\theta + v_y y_\theta = -v_x r \sin\theta + v_y r\cos\theta \end{aligned}$$

The Cauchy–Riemann equations then become

$$u_x = v_y, \qquad u_y = -v_x \iff ru_r = v_\theta, \qquad u_\theta = -rv_r.$$

12. January 31

Theorem 12.1. *Let $f(z) = u(r,\theta)+iv(r,\theta)$ be defined in some neighborhood of $z_0 = r_0 e^{i\theta_0} \neq 0$. Suppose that*

(1) u_r, u_θ, v_r, v_θ exist and are continuous everywhere in the NBHD, and
(2) these partial derivatives satisfy the Cauchy–Riemann equations at (r_0, θ_0).

Then $f'(z_0)$ exists and equals $e^{-i\theta_0}(u_r(r_0,\theta_0) + iv_r(r_0,\theta_0))$.

Definition 12.2. *A function $f(z)$ is* **holomorphic in an open set** *S if $f'(z)$ exists for each $z \in S$, and $f(z)$ is* **holomorphic at a point** *z_0 if it is holomorphic in some NBHD of z_0. Finally, $f(z)$ is* **entire** *if f is holomorphic at each point in $\mathbb{C}$.*

Sums, products, quotients, and compositions of analytic functions are holomorphic.

Theorem 12.3. *If $f : U \to \mathbb{C}$ is holom. on U and $f' = 0$ for all $z \in U$, then $f =$ const.*

Proof sketch. Cauchy–Riemann equations and multivariable mean value theorem □

If f is not analytic at z_0 but analytic in a deleted NBHD of z_0, then z_0 is called a **singular point**, or **singularity**, of f. Singularities of functions will play an important role later on.

Let $U \subseteq \mathbb{R}^2$. A function $H : U \to \mathbb{R}$ is **harmonic** on U if it has continuous partial derivatives of the first and second order and satisfies **Laplace's equation**

$$H_{xx}(x,y) + H_{yy}(x,y) = 0.$$

Example 12.4. Let $H(x,y) = x/(x^2+y^2)$. Clearly, the first and second order partial derivatives will be continuous when $x^2+y^2 \neq 0$. We have $H_{xx} + H_{yy} = 2x(x^2-3y^2)/(x^2+y^2)^3 + (-2x(x^2-3y^2)/(x^2+y^2)^3) = 0$

Theorem 12.5. *If $f(z) = u + iv$ is holom. on $D \subseteq \mathbb{C}$, then u, v are harmonic on D.*

We will assume a result to be proven later on (Sec. 57) that if $f : D \to \mathbb{C}$ is holomorphic on D, then the partial derivatives of first and second orders for u and v are continuous.

Proof. If f is analytic on D, then u and v satisfy the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x.$$

Differentiating both equations with respect to x and y, we arrive at

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}, \qquad u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}.$$

Since the partial derivatives are assumed to be continuous, we have that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. Hence $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$ on D. □

We will later show that every harmonic function is the real part of a holomorphic function.

Definition 12.6. *We define $e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)$. (justified later)*

In polar form, we have $e^z = \rho e^{i\phi}$, where $\rho = e^x$ and $\phi = y$. Thus it is clear that $|e^z| = e^{\mathrm{Re}(z)}$ and $\arg(e^z) = \{y + 2n\pi \colon n \in \mathbb{Z}\}$. Since $e^x \neq 0$, we have

$$e^z \neq 0, \qquad z \in \mathbb{C}.$$

It is now clear, since $e^{x+y} = e^x e^y$ and we already proved that $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$, we have that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

Since $e^{z_1-z_2}e^{z_2} = e^{z_1}$, we find that

$$e^{z_1-z_2} = e^{z_1}/e^{z_2}, \qquad 1/e^z = e^{-z}, \qquad e^0 = 1.$$

Hence if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ with $r_1 = r_2$ and there exists an integer n such that $\theta_1 - \theta_2 = 2\pi n$, then $e^{z_1} = e^{z_2}$.

We computed (earlier example) that $\frac{d}{dz}e^z = e^z$. Since this holds on all of $\mathbb{C}$, e^z is entire.

Note that if $n \in \mathbb{Z}$, then

$$e^{z+2\pi in} = e^z e^{2\pi in} = e^z,$$

so e^z is $2\pi i$-periodic, unlike e^x for $x \in \mathbb{R}$. Moreover, $e^{i\pi} = -1$, whereas $e^x \geq 0$ for all $x \in \mathbb{R}$.

13. February 3

If $0 \neq w = re^{i\theta}$, where $\theta = \Theta + 2\pi n$ for some $n \in \mathbb{Z}$, then we can always solve the equation

$$e^z = w, \qquad z = \ln r + i(\Theta + 2\pi n), \qquad n \in \mathbb{Z}.$$

Definition 13.1. *If $0 \neq z = re^{i\theta}$, then we define the multivalued function $\log z = \ln r + i(\theta + 2\pi n)$, $n \in \mathbb{Z}$. We abbreviate this as $\log z = \ln |z| + i \arg(z)$.*

Hence if $0 \neq z = re^{i\theta}$ with $\theta \in \arg(z)$, then

$$e^{\log z} = e^{\ln r} e^{i(\theta + 2\pi n)} = re^{i\theta} = z.$$

Note that if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$\log(z_1 z_2) = \log(r_1 r_2 e^{i(\theta_1 + \theta_2)}) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2\pi n), \qquad n \in \mathbb{Z},$$

which differs from $\log z_1 + \log z_2$ by an integer multiple of $2\pi i$.

On the other hand, if $z = x + iy$, then for any $n \in \mathbb{Z}$.

$$\log e^z = \log(e^{x+iy}) = \ln e^x + i(y + 2\pi n) = z + 2\pi i n$$

If we let $\alpha \in \mathbb{R}$, then for $\theta \in (\alpha, \alpha + 2\pi)$, the function

$$\log : \{z \in \mathbb{C} \colon z = |z|e^{i\theta},\ |z| > 0,\ \theta \in (\alpha, \alpha + 2\pi)\} \to \mathbb{C} \colon \log z = \ln |z| + i\theta$$

is *single-valued* and continuous, even holomorphic (since it satisfies the Cauchy–Riemann equations $ru_r = v_\theta$ and $u_\theta = -rv_r$). It would be single-valued but not continuous if we included the ray $\theta = \alpha$. The polar form of the derivative is

$$\frac{d}{dz} \log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(r^{-1} + i0) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Taking $\alpha = 0$, we have

$$\frac{d}{dz} \text{Log } z = \frac{1}{z}, \qquad |z| > 0, \qquad -\pi < \text{Arg } z < \pi.$$

We obtain different functions for each choice of α. We need to distinguish these functions.

Definition 13.2. *A* **branch** *of a multivalued function f is a single-valued function F that is holomorphic in some domain at each z, of which the value $F(z)$ is one of the values of f.*

For any given $\alpha \in \mathbb{R}$, we obtain a different branch of the log function. But the one that matters to us the most corresponds with our definition of $\text{Arg}(z)$, which is the argument θ lying in the interval $-\pi < \theta \leq \pi$. So the **principal branch** of the log function is

$$\text{Log } z = \ln |z| + i\text{Arg}(z), \qquad |z| > 0.$$

A **branch cut** is a portion of a line or curve that is introduced in order to define a branch F of a multivalued function. The principal branch cut for log consists of the origin and the ray $\Theta = \pi$.

Example 13.3. The branch cut corresponding with $\alpha = -\pi$ gives the **principal value** of log, namely

$$\text{Log } z = \ln |z| + i\text{Arg}(z).$$

of the log function is the ray $\theta = \pi$.

Since $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$, we have $\log z_1z_2 = \log z_1 + \log z_2$. (EMPHASIZE: If you specify any two of the three arguments, then the third argument is uniquely determined!)

But since $\text{Arg}(z_1z_2)$ does not generally equal $\text{Arg}(z_1)+\text{Arg}(z_2)$, Log z_1z_2 does not generally equal Log $z_1 +$ Log z_1.

Example 13.4. $\log(-1) = i\pi + 2\pi in$, $n \in \mathbb{Z}$. $\text{Log}(-1) = \pi i$.

Example 13.5. The two square roots of i are $e^{\pi i/4}$ and $e^{5\pi i/4}$. We have

$$\log(e^{\pi i/4}) = i(\pi/4 + 2\pi n), \qquad n \in \mathbb{Z}$$

and

$$\log(e^{5\pi i/4}) = i(\pi/4 + (2n+1)\pi), \qquad n \in \mathbb{Z}.$$

Hence

$$\log(i^{1/2}) = i(\pi/4 + \pi n).$$

Also,

$$\frac{1}{2}\log i = \frac{1}{2}(i\pi/2 + 2\pi n) = (i\pi/4 + n\pi), n \in \mathbb{Z}.$$

Hence $\log(i^{1/2}) = \frac{1}{2}\log i$.

Example 13.6. Note that $\log(i^2) = \log(-1) = i\pi + 2\pi in$, $n \in \mathbb{Z}$. But $2\log i = 2(\frac{\pi}{2} + 2\pi in) = i\pi + 4\pi in$, $n \in \mathbb{Z}$. So $\log(i^2) \neq 2\log i$, **unless one specifies a suitable branch**. On the other hand, if we write $2\log i = i(\pi/2 + 2\pi m) + i(\pi/2 + 2\pi n)$, then we are consistent with $\log z_1z_2 = \log z_1 + \log z_2$ (in the same sense that $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$).

Example 13.7. If $\text{Re}(z_1), \text{Re}(z_2) > 0$, then $z_1 = |z_1|\exp(i\Theta_1)$ and $z_2 = |z_2|\exp(i\Theta_2)$ with $\Theta_1, \Theta_2 \in (-\pi/2, \pi/2)$. Since $-\pi < \Theta_1 + \Theta_2 < \pi$, we have that

$$\text{Arg}(z_1z_2) = \Theta_1 + \Theta_2,$$

so

$$\text{Log}(z_1z_2) = \ln|z_1z_2| + i\text{Arg}(z_1z_2) = \ln|z_1| + \ln|z_2| + i(\text{Arg}(z_1) + \text{Arg}(z_2)) = \text{Log } z_1 + \text{Log } z_2.$$

Let $z \neq 0$. Since $e^{\log z} = z$, one can prove inductively that for any integer n, we have $e^{n\log z} = z^n$. Similarly, one can show (taking n-th roots as we described a while back) that for each integer $n \geq 1$, we have

$$e^{\frac{1}{n}\log z} = e^{\frac{1}{n}\log|z| + \frac{i(\Theta+2\pi k)}{n}} = |z|^{1/n}\exp\Big(\frac{i(\Theta+2\pi k)}{n}\Big), \qquad k \in \mathbb{Z},$$

which equals $z^{1/n}$.

Definition 13.8. *For $c \in \mathbb{C}$, we define $z^c = e^{c\log z}$ for $z \neq 0$.*

Note that since $e^{-z} = 1/e^z$, we have $1/z^c = 1/e^{c\log z} = e^{-c\log z} = z^{-c}$. Since $e^{\log z} = z$ always, on a branch of $\log z$, we can compute via the chain rule

$$\frac{d}{dz}z^c = \frac{d}{dz}e^{c\log z} = e^{c\log z}\frac{c}{z} = c\frac{e^{c\log z}}{e^{\log z}} = ce^{(c-1)\log z} = cz^{c-1}.$$

The principal value of z^c is given by the branch of log corresponding with Log:

$$z^c = e^{c\text{Log } z}.$$

This corresponds to the **principal branch** of z^c on the domain $|z| > 0$, $-\pi < \text{Arg } z < \pi$.

14. February 5

We can also define, for $c \neq 0$,
$$c^z = e^{z \log c},$$
and when a value of $\log c$ is specified, then c^z is an entire function of z with
$$\frac{d}{dz} e^{z \log c} = e^{z \log c} \log c = z^c \log c.$$

Example 14.1.
$$i^i = e^{i \log i} = e^{i(i(\pi/2 + 2\pi n))} = e^{-\pi/2 + 2\pi n}, \qquad n \in \mathbb{Z}.$$
Principal value ($n = 0$): $i^i = e^{-\pi/2}$.

Trig functions

Recall that $e^{i\theta} = \cos\theta + i \sin\theta$, and
$$\mathrm{Re}(e^{i\theta}) = \frac{e^{i\theta} + \overline{e^{i\theta}}}{2} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$
and
$$\mathrm{Im}(e^{i\theta}) = \frac{e^{i\theta} - \overline{e^{i\theta}}}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2} = \sin\theta$$

Definition 14.2.
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \tan z = \frac{\sin z}{\cos z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}, \qquad \cot z = \frac{\cos z}{\sin z}.$$

It is straightforward to verify from our definitions that
$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z), \qquad e^{iz} = \cos z + i \sin z,$$
From
$$\frac{d}{dz} e^{cz} = c e^{cz}, \qquad c = \pm i,$$
we find that
$$\frac{d}{dz} \sin z = \cos z, \qquad \frac{d}{dz} \cos z = -\sin z.$$
All of the usual trigonometric identities from real variables carry over:
$$\sin^2 z + \cos^2 z = 1, \qquad \sin 2z = 2 \sin z \cos z, \cos 2z = \cos^2 z - \sin^2 z, \ldots.$$

15. February 7

Definition 15.1.

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$
$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}, \qquad \operatorname{sech}(z) = \frac{1}{\cosh(z)}, \qquad \coth(z) = \frac{1}{\tanh(z)}.$$

Since

$$\sin(z_1 z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1, \qquad \cos(z_1 z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

we have

$$\sin(x+iy) = \sin(x)\cos(iy) + \sin(iy)\cos(x), \qquad \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy).$$

Now,

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = i\sinh(y), \qquad \cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \cosh(y).$$

Thus

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y), \qquad \cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

Since $\cos^2 t + \sin^2 t = 1$, we find that

$$|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y.$$

Hence $\cos z$ and $\sin z$ are UNBOUNDED on $\mathbb{C}$, unlike $\cos x$ and $\sin x$ on $\mathbb{R}$.

Theorem 15.2. *We have $\sin z = 0$ if and only if $z = n\pi$, where $n \in \mathbb{Z}$. We have $\cos z = 0$ if and only if $z = \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$.*

Proof. For $\sin(z)$, set the absolute values equal to zero and solve for x, y. You could do the same for $\cos(z)$, or note that $\cos(z) = -\sin(z - \frac{\pi}{2})$. ☐

This tells us the zeros and singularities of $\tan(z)$, $\cot(z)$, $\csc(z)$, $\sec(z)$. Also, using the quotient rule and the above trig identities,

$$(\tan z)' = \sec^2 z, \quad (\cot z)' = -\csc^2 z, \quad (\sec z)' = \sec z \tan z, \quad (\csc z)' = -\csc z \cot z.$$

Hyperbolic trig functions will also come up. Since $(e^z)' = e^z$, we have by the chain rule

$$(\sinh z)' = \cosh z, \qquad (\cosh z)' = \sinh z.$$

From our definition, we have

$$-i\sinh(iz) = \sin z, \quad -i\sin(iz) = \sinh(z), \quad \cosh(iz) = \cos z, \quad \cos(iz) = \cosh(z).$$

Since $\cos(z)$ and $\sin(z)$ are 2π-periodic, it follows that $\cosh(z)$ and $\sinh(z)$ are $2\pi i$-perodic.

The book has a list of frequently used hyperbolic trig identities which look similar to (but are different from!) the usual trig identities. For example:

$$\cos^2(z) - \sin^2(z) = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z})}{4} = 1.$$

Theorem 15.3. *We have $\sinh(z) = 0$ precisely when $z = n\pi i$, $n \in \mathbb{Z}$. We have $\cosh z = 0$ if and only if $z = (\frac{\pi}{2} + n\pi)i$, $n \in \mathbb{Z}$.*

Theorem 15.4. *I leave it as an exercise to verify that* $|\sinh(z)|^2 = \sinh^2 x + \sin^2 y$ *and* $|\cosh z|^2 = \sinh^2 x + \cos^2 y$. *Set the moduli equal to zero and solve for* x, y.

I also leave it as an exercise to compute

$(\tanh z)' = \text{sech}^2 z, \quad (\sec z)' = -\text{sech} z \tanh z, \quad (\text{coth} z)' = -\text{csch}^2 z, \quad (\text{csch} z)' = -\text{csch} z \text{coth} z.$

Now, let us solve the equation

$$z = \sin w, \qquad (w = \arcsin z).$$

for w. Recall that

$$z = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}} \implies e^{2iw} - 2ie^{iw}z - 1 = 0.$$

This is solved via the quadratic formula:

$$e^{iw} = iz + \sqrt{1 - z^2},$$

where we think of the square root function as multivalued. Taking logs, we conclude that

$$\arcsin z = -i \log(iz + \sqrt{1 - z^2}).$$

Similarly, we compute

$$\arccos z = -i \log(z + i\sqrt{1 - z^2}), \qquad \arctan z = \frac{i}{2} \log \frac{i + z}{i - z}.$$

These are all multivalued. On specific branches, these become single-valued holomorphic. We can compute

$$(\arctan z)' = \frac{1}{z^2 + 1}.$$

This is single-valued. But the derivatives

$$(\arcsin z)' = \frac{1}{\sqrt{1 - z^2}}, \qquad (\arccos z)' = -\frac{1}{\sqrt{1 - z^2}}$$

are multivalued.

We also can solve

$$z = \cosh w \qquad (w = \text{arccosh} z)$$

Then

$$z = \frac{e^w + e^{-w}}{2} = \frac{e^{2w} + 1}{2e^w} \implies e^{2w} - 2e^w z + 1 = 0.$$

Again, by the quadratic formula and taking logs,

$$\text{arccosh} z = \log(z + \sqrt{z^2 - 1}).$$

Similarly,

$$\text{arcsinh} z = \log(z + \sqrt{z^2 + 1}), \qquad \text{arctanh} z = \frac{1}{2} \log \frac{1 + z}{1 - z}.$$

16. February 10

We begin to set up integration for a function of a complex variable.

16.1. **Complex functions on a real interval.** If $f(t) = u(t)+iv(t)$ is continuous on (a, b), we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

This very closely mimics the real integral. If $c = \alpha + i\beta \in \mathbb{C}$, then

$$\int_a^b cf dt = \int_a^b (\alpha u - \beta v + i(\alpha v + \beta u)dt = \int_a^b (\alpha u - \beta v)dt + i\int_a^b (\alpha v + \beta u)dt = c\int_a^b f dt,$$

and, with $\theta = \arg \int_a^b f(t)dt$, we find that

$$\Big|\int_a^b f(t)dt\Big| = \mathrm{Re}\Big(e^{-i\theta}\int_a^b f(t)dt\Big) = \int_a^b \mathrm{Re}(e^{-i\theta} f(t))dt \le \int_a^b |f(t)|dt.$$

16.2. **Complex functions on an arc.** Let γ be a piecewise differentiable arc with equation $z(t)$, $a \le t \le b$. If f is defined and continuous on γ, then $f(z(t))$ is also continuous and we can define

$$(***)\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

This is our *definition* of the complex line integral of $f(z)$ extended over the arc γ. On the RHS of $(***)$, if $z'(t)$ is not continuous throughout, then the interval of integration is subdivided into subintervals along which z' is continuous. We will tacitly assume that γ is piecewise differentiable.

The most important property of $(***)$ is its invariance under a change of parameter: If $t = t(\tau)$ is increasing and maps $\alpha \le \tau \le \beta$ onto $a \le t \le b$ and τ is piecewise differentiable, then

$$\int_a^b f(z(t))z'(t)dt = \int_\alpha^\beta f(z(t(\tau)))z'(t(\tau))t'(\tau)d\tau = \int_\alpha^\beta f(z(t(\tau)))\frac{d}{d\tau}z(t(\tau))d\tau.$$

If $-\gamma$ is defined by $z = z(-t)$, $-b \le t \le -a$, then

$$\int_{-\gamma} f(z)dz = \int_{-a}^{-b} f(z(-t))(-z'(-t))dt = \int_b^a f(z(t))z'(t)dt = -\int_\gamma f(z)dz.$$

If $\gamma = \gamma_1 + \cdots \cup \gamma_n$ is a subdivision of γ into disjoint subarcs, then

$$\int_\gamma = \sum_{j=1}^n \int_{\gamma_j}$$

Definition 16.1. *An arc γ (or C in the book) is a* **simple (Jordan) arc** *if it does not cross itself ($z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$). When C is simple except for $z(a) = z(b)$, then C is a* **simple closed (Jordan) curve**. *Such a curve is* **positively oriented** *when it is in the counterclockwise direction.*

The integral over a closed curve is also invariant under a shift of parameter: The old and new initial points determine two subarcs γ_1, γ_2, and the invariance follows from the fact that $\int_{\gamma_1\cup\gamma_2} = \int_{\gamma_1} + \int_{\gamma_2}$.

One of the most commonly used arcs in the integrals we compute will be the closed arc

$$z(t) = e^{2\pi i t}, \qquad 0 \leq t \leq 1.$$

This is an example of a positively oriented Jordan curve. It is different from

$$z(t) = e^{-2\pi i t}, \qquad 0 \leq t \leq 1$$

because this is not positively oriented. It is also different from

$$z(t) = e^{2\pi i t}, \qquad 0 \leq t \leq 2$$

since this traverses the same path *twice*.

We can also consider the line integral with respect to $\overline{z}$:

$$\int_\gamma f(z) d\overline{z} = \overline{\int_\gamma \overline{f(z)} dz}.$$

Using this notation, we can introduce line integrals with respect to $x = \text{Re}(z)$ or $y = \text{Im}(z)$:

$$\int_\gamma f(z) dx = \frac{1}{2}\Big(\int_\gamma f(z) dz + \int_\gamma f(z) d\overline{z}\Big), \qquad \int_\gamma f(z) dy = \frac{1}{2i}\Big(\int_\gamma f(z) dz - \int_\gamma f(z) d\overline{z}\Big).$$

Hence if $f = u + iv$, then

$$\int_\gamma f dz = \int_\gamma (u dx - v dy) + i \int_\gamma (u dy + v dx).$$

17. February 12

Midterm Exam 1

18. February 14

An essentially different line integral is obtained by integration with respect to **arclength**:

$$\int_\gamma f ds = \int_\gamma f|dz| = \int_\gamma f(z(t))|z'(t)|dt.$$

By proceeding as earlier, the integral is again independent of the choice of parameter. But, unlike before,

$$\int_{-\gamma} f(z)|dz| = \int_\gamma f(z)|dz|.$$

Also,

$$\Big|\int_a^b f(t)dt\Big| \le \int_a^b |f(t)|dt \implies \Big|\int_\gamma f(z)dz\Big| \le \int_\gamma |f(z)|\cdot|dz|.$$

If $f \equiv 1$, then

$$\int_\gamma f(z)|dz| = \int_\gamma |dz| = \text{length of } \gamma$$

Example 18.1. Let $z(t) = a + \rho e^{it}$, $0 \le t \le 2\pi$. This traces the circle of radius ρ centered at $z = a$. Then $z'(t) = i\rho e^{it}$, and

$$\int_\gamma |dz| = \int_0^{2\pi} |z'(t)|dt = \int_0^{2\pi} \rho dt = 2\pi\rho.$$

Example 18.2. Let $z(t) = e^{2\pi it}$, $0 \le t \le 1$. Then

$$\int_\gamma \frac{dz}{z} = \int_0^1 \frac{2\pi i e^{it}}{e^{2\pi it}}dt = 2\pi i\int_0^1 dt = 2\pi i.$$

Let $n \ge 1$ be an integer. We could also consider $z(t) = e^{2\pi it}$, $0 \le t \le n$ (corresponding to n rotations around the circle). Then the above integral becomes $2\pi in$.

Example 18.3. Let γ be a piecewise differentiable curve from z_1 to z_2 traced out by $z(t)$, $a \le t \le b$. Then

$$\int_\gamma zdz = \int_a^b z(t)z'(t)dt = \frac{z(t)^2}{2}\bigg|_a^b = \frac{z(b)^2 - z(a)^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Thus the integral depends only on the endpoints of γ.

Example 18.4. Let γ be given by $z = 3e^{it}$, $0 \le t \le \pi$. This traces a semicircular parth from $z = 3$ to $z = -3$. Choose the branch

$$f(z) = \sqrt{z} = e^{\frac{1}{2}\log z}, \qquad |z| > 0, 0 < \arg z < 2\pi.$$

Then

$$\int_\gamma \sqrt{z}dz = \int_0^\pi \sqrt{3}e^{it/2}\cdot 3ie^{it}dt = 3^{3/2}i\int_0^\pi e^{3it/2}dt = 3^{3/2}i\cdot -\frac{2}{3i}e^{3it/2}\bigg|_0^\pi = -2\sqrt{3}(1+i).$$

Example 18.5. Let γ be the curve traced by $z(t) = e^{it}$, $-\pi \le t \le \pi$. Using the principal branch

$$f(z) = z^{i-1} = \exp[(i-1)\text{Log}z], \qquad |z| > 0, -\pi < \text{Arg}z < \pi,$$

we compute

$$\int_\gamma z^{i-1}dz = \int_{-\pi}^{\pi} e^{(i-1)it} ie^{it}dt = i\int_{-\pi}^{\pi} e^{-t}dt = -ie^{-t}\Big|_{-\pi}^{\pi} = -i(e^{-\pi}-e^{\pi}) = 2i\cdot\frac{e^{\pi}-e^{-\pi}}{2} = 2i\sinh(\pi).$$

Example 18.6. Let C be parametrized by $z(t) = 2e^{it}$, $0 \leq t \leq \pi/2$. Then for z on C, we have $|z-2| \leq |z| + 2 = 4$ and $|z^4+1| \geq ||z^4|-1| = 15$ by the triangle inequality. Thus

$$\Big|\int_C \frac{z-2}{z^4+1}dz\Big| \leq \int_C \Big|\frac{z-2}{z^4+1}\Big| \cdot |dz| \leq \frac{4}{15}\int_C |dz| = \frac{4}{15}\int_0^{\pi/2} |z'(t)|dt = \frac{4}{15}\int_0^{\pi/2} 2dt = \frac{4\pi}{15}.$$

We could be more precise: $|z-2| = 2|e^{it}-1|$, which is maximized as a function of t when $t = \pi/2$, in which case $|z-2| \leq \sqrt{2} < 4$.

More generally, if ℓ is the arclength of C and $|f(z)| \leq M$ for all $z \in C$, then

$$\Big|\int_C f dz\Big| \leq \int_C |f| \cdot |dz| \leq M\ell.$$

19. February 17

In our example with $\int_\gamma z dz$, we saw that the integral only depended on the endpoints of the arc γ. When else might this be true?

Theorem 19.1. *The line integral $\int_\gamma (pdx + qdy)$, defined on a region D, depends only on the endpoints of γ if and only if there exists a function $F(x, y)$ on D with partial derivatives $F_x = p$, $F_y = q$.*

Proof. If $F_x = p$, $F_y = q$, then

$$\int_\gamma (pdx+qdy) = \int_a^b \Big(F_x x'(t)+F_y y'(t)\Big)dt = \int_a^b \frac{d}{dt}F(x(t), y(t))dt = F(x(b), y(b))-F(x(a), y(a)).$$

On the other hand, if the integral depends only on the endpoints $z_0 = (x_0, y_0)$ and $z_1 = (x, y)$, we can define

$$F(x, y) = \int_\gamma (pdx' + qdy').$$

then we can take γ to be piecewise differentiable with the pieces running parallel to the coordinate axes. Along the vertical component, $dy' = 0$. Thus

$$F(x, y) = \int_{x_0}^{x} p(x', y')dx' + const.$$

Then by the fundamental theorem of calculus, $F_x = p$. Along the vertical component, $dy' = 0$. Thus

$$F(x, y) = \int_{y_0}^{y} q(x', y')dy' + const.$$

Then by the fundamental theorem of calculus, $F_y = q$. □

If the expression $pdx + qdy$ can be written in the form

$$dU = U_x dx + U_y dy,$$

then $f(z) = U_x + iU_y$ is the derivative of a function $U(z)$. Thus an integral depends only on the endpoints if and only if the integral is an exact differential. Observe that p, q, U can be either real or complex. The function U, if it exists, is unique up to an additive constant (if two functions have the same partial derivatives, then their difference must be constant).

Let f be continuous. When is $f(z)dz = f(z)dx+if(z)dy$ an exact differential? According to the definition, there must be a function $F(z) = u(x, y) + iv(x, y)$ in D with partial derivatives

$$F_x(z) = f(z), \qquad F_y(z) = if(z).$$

Then

$$u_x + iv_x = f, \qquad u_y + iv_y = if \implies u_x + iv_x = f = v_y - iu_y.$$

Thus

$$u_x = v_y, \qquad u_y = -v_x,$$

which are the Cauchy–Riemann equations. This leads us to:

Theorem 19.2. *Let f be continuous. The integral $\int_C f dz$ depends on the endpoints of γ if and only if f is the derivative of a holomorphic function on D.*

Example 19.3. Let $k \geq 2$ be an integer. The function $f(z) = z^{-k}$ is continuous everywhere except $z = 0$, and outside of that point, f is the derivative of $\frac{1}{1-k} z^{1-k}$. Thus if C is parametrized by $z(t) = e^{it}$, $-\pi \leq t \leq \pi$, then

$$\int_C \frac{dz}{z^2} = 0.$$

This *fails* when $k = 1$ because the antiderivative of any branch $F(z)$ of $\log z$ (the antiderivative of $1/z$) is not defined along its branch cut. So C does not lie in any domain throughout which $F'(z) = 1/z$, and so one cannot make use of an antiderivative. Fortunately, we could evaluate the $k = 1$ case directly by parametrization.

20. February 19

Recall Green's theorem from multivariable calculus: If D is a region bounded by a closed curve C and L, M are functions defined on an open region containing D and having continuous partial derivatives there, then

$$\int_C (Ldx + Mdy) = \int\int_D (M_x - L_y)dxdy,$$

where C is positively oriented. Now, suppose that $f(z) = u(x, y) + iv(x, y)$. Then

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy).$$

By Green's theorem, if the first order partial derivatives are continuous, then f being holomorphic implies, via the Cauchy–Riemann equations, that

$$\int_C f(z)dz = \int\int_D (-v_x - u_y)dxdy + i\int_C (u_x - v_y)dxdy = 0.$$

We have thus concluded a preliminary version of Cauchy's theorem.

Theorem 20.1. *If f is analytic on a region R with piecewise differentiable boundary ∂R, and f' is continuous there, then*

$$\int_{\partial R} fdz = 0.$$

We are now on track to prove the **Cauchy–Goursat Theorem**:

Theorem 20.2. *If f is analytic at all points interior to and on a simple closed curve C, then $\int_C f(z)dz = 0$.*

This removes the condition that f' is continuous that we needed when we appealed only to Green's theorem. We begin with an auxiliary result of Goursat.

Theorem 20.3 (Goursat)**.** *Let T be a triangular region, and let f be holomorphic on T (means f is complex-differentiable on an open set containing T), then*

$$\int_{\partial T} f(z)dz = 0.$$

Proof. Let

$$M = \Big|\int_{\partial T} f(z)dz\Big|, \qquad \ell = \text{ perimeter of } \partial T.$$

Our goal is to show that $M = 0$.

Step 1: Divide and conquer. Bisect the sides of T, dividing T into 4 triangles $T_1, \dots, T_4$. Orient each sub-triangle the same way as T. After cancelling segments, we have

$$\sum_{j=1}^{4}\int_{\partial T_j} f(z)dz = \int_{\partial T} f(z)dz.$$

Choose a rectangle T_j, say T^1, such that $|\int_{\partial T_j} fdz|$ is maximized. Then by the triangle inequality for integrals,

$$M \le \sum_{j=1}^{4}\Big|\int_{\partial T_j} f(z)dz\Big| \le 4\Big|\int_{\partial T^1} f(z)dz\Big|.$$

Then

$$\Big|\int_{\partial T^1} f(z)dz\Big| \geq \frac{M}{4}.$$

Step 2: Get to the limit point z^*. The argument above for T can be repeated for T^1, which produces a rectangle $T^1 \supsetneq T^2$ such that

$$\Big|\int_{\partial T^2} f(z)dz\Big| \geq \frac{M}{4^2}.$$

Proceeding inductively, we obtain a sequence of triangles (T^n) such that

$$T^1 \supsetneq T^2 \supsetneq \cdots \supsetneq T^n \supsetneq \cdots, \qquad \Big|\int_{\partial T^n} f dz\Big| \geq \frac{M}{4^n}, \qquad \text{perimeter}(T^n) = \frac{\ell}{2^n}.$$

Since each rectangle is contained in the former and their diameters are tending to zero as $n \to \infty$ $(diam/2^n)$, we can conclude that there exists a unique point z^* contained in every T^n.

Step 3: Use differentiability and squeeze. Let $\varepsilon > 0$ be arbitrary. Since f is holomorphic at $z^* \in T$, there exists $\delta > 0$ such that if $0 < |z - z^*| < \delta$, then

$$\Big|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\Big| < \varepsilon, \qquad \text{hence} \qquad |f(z) - f(z^*) - f'(z^*)(z - z^*)| < \varepsilon|z - z^*|.$$

We have already proved that

$$\int_{\partial T^n} 1 dz = \int_{\partial T^n} z dz = 0,$$

so, upon expanding,

$$\begin{aligned}&\int_{\partial T^n} (f(z) - f(z^*) - f'(z^*)(z - z^*))dz \\ &= \int_{\partial T^n} f dz - (f(z^*) + f'(z^*)) \int_{\partial T^n} 1 dz - f'(z^*) \int_{\partial T^n} z dz = \int_{\partial T^n} f dz.\end{aligned}$$

Now, if we choose $n > \log_2(\ell/\delta)$, then perimeter$(T^n) < \delta$. Hence if $z \in T^n$, then

$$|z - z^*| < \text{perimeter}(T^n) = \frac{\ell}{2^n} < \delta.$$

Hence

$$\begin{aligned} 0 \leq \frac{M}{4^n} \leq \Big|\int_{\partial T^n} f dz\Big| &= \Big|\int_{\partial T^n} (f(z) - f(z^*) - f'(z^*)(z - z^*))dz\Big| \\ &\leq \int_{\partial T^n} |(f(z) - f(z^*) - f'(z^*)(z - z^*))| \cdot |dz| \\ &\leq \int_{\partial T^n} \varepsilon|z - z^*| \cdot |dz| \leq \int_{\partial T^n} \varepsilon \frac{\ell}{2^n}|dz| \\ &= \varepsilon \frac{\ell}{2^n}\frac{\ell}{2^n} = \frac{\varepsilon \ell^2}{4^n}. \end{aligned}$$

Thus $0 \leq M \leq \varepsilon\ell^2$. But since $\varepsilon > 0$ was arbitrary, we can squeeze by letting $\varepsilon \to 0$ and conclude that $M = 0$. □

21. February 21

We will apply Goursat's theorem to prove the following.

Theorem 21.1. *If f is a holomorphic function on an open disc D, then there exists a holomorphic function F on D such that $F' = f$ on D.*

Proof. Translate D so that it is centered at $z = 0$. For $z \in D$, let ℓ_z be the straight line path from 0 to z. Clearly, $\ell_z \subset D$ (this is why D is a disc!). Define

$$F(z) = \int_{\ell_z} f(w)dw = \int_0^z f(w)dw.$$

Goal: Show that F is holomorphic on D with $F' = f$.

Let $h \in D$ be such that $z + h \in D$. Note that

$$\int_0^{z+h} f(w)dw + \int_{z+h}^{z} f(w)dw + \int_z^0 f(w)dw = 0$$

by Goursat's theorem. Hence

$$\int_z^{z+h} f(w)dw = \int_0^{z+h} f(w)dw - \int_0^z f(w)dw = F(z+h) - F(z).$$

Now,

$$\int_z^{z+h} f(w)dw = \int_z^{z+h} f(z)dw + \int_z^{z+h}(f(w) - f(z))dw = f(z)h + \int_z^{z+h}(f(w) - f(z))dw.$$

By the triangle inequality and our "triangle inequality for integrals",

$$\begin{aligned}
\Big|\frac{F(z+h) - F(z)}{h} - f(z)\Big| &= \Big|\frac{1}{h}\int_z^{z+h}(f(w) - f(z))dw\Big| \\
&\leq \frac{1}{|h|}\int_z^{z+h}|f(w) - f(z)| \cdot |dw| \\
&\leq \frac{h}{h}\max_{|z-w|\leq|h|}|f(w) - f(z)| = \max_{|z-w|\leq|h|}|f(w) - f(z)|.
\end{aligned}$$

By the continuity of f (since f is holomorphic), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|h| < \delta$, then for each w such that $|z - w| \leq |h|$,

$$|f(w) - f(z)| < \varepsilon.$$

Thus

$$|h| < \delta \implies \Big|\frac{F(z+h) - F(z)}{h} - f(z)\Big| < \varepsilon.$$

In other words, $F'(z) = f(z)$. □

Recall that we proved:

Theorem 21.2. *Let f be continuous on D, and let $C \subset D$. The integral $\int_C f dz$ depends on the endpoints of C if and only if f is the derivative of a holomorphic function on D.*

This allows us to conclude the Cauchy–Goursat theorem.

Theorem 21.3 (Cauchy–Goursat)**.** *If f is holomorphic on a disc D containing a closed curve γ, then $\int_\gamma f dz = 0$.*

In the Miscellany, I will describe how one moves from D a disc to any open set.

A **simply-connected** domain D is a domain such that every simple closed contour within it contains only points of D.

Theorem 21.4. *If f is holomorphic and throughout a simply connected D and C is a closed contour in D, then*

$$\int_C f(z)dz = 0.$$

Proof. If C intersects itself at most finitely many times, then the proof follows easily from Cauchy–Goursat. See Ex. 5, Sec. 53 for one strategy when C intersects itself infinitely many times. □

Corollary 21.5. *If f is holomorphic throughout a simply connected domain D, then it must have an antiderivative everywhere in D.*

Proof. The antiderivative is constructed as in our proof of Cauchy–Goursat. □

Corollary 21.6. *Entire functions always possess antiderivatives.*

Proof. Since $\mathbb{C}$ is simply connected, this follows from the previous corollary. □

22. February 24

A domain that is not simply connected is **multiply connected**.

Theorem 22.1. *Suppose that*

1. *C is a simple closed contour, positive orientation.*
2. *C_k, $1 \le k \le n$, are simple closed contours interior to C, each with* **negative** *orientation, that are disjoint and whose interiors have no points in common.*

If f is holomorphic on all of these contours and throughout the multiply connected domain of $\mathrm{int}(C) - \cup_k \mathrm{int}(C_k)$, then

$$\int_C f\,dz + \sum_{k=1}^{n} \int_{C_k} f\,dz = 0.$$

Proof. Draw the picture. □

Corollary 22.2 (Principle of deformation of paths)**.** *Let C_1, C_2 be positively oriented simple closed contours with C_1 interior to C_2. (Draw picture.) If f is holomorphic in the closed region consisting of those contours and all points between them, then*

$$\int_{C_1} f\,dz = \int_{C_2} f\,dz.$$

Proof. Follows from previous theorem:

$$\int_{C_2} + \int_{-C_1} = 0.$$

□

Example 22.3. We proved earlier that

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i.$$

Now, we find that if γ is any simple closed curve containing the origin, then

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

By a similar parametrization, we have

$$\int_{|z-a|=1} \frac{dz}{z-a} = 2\pi i,$$

and any simple closed curve γ containing a satisfies

$$\int_\gamma \frac{dz}{z-a} = 2\pi i.$$

We now come to the Cauchy Integral Formula, another landmark result in complex analysis.

Theorem 22.4. *Let f be holomorphic inside and on a simple closed contour C, positively oriented. If z_0 is in the interior of C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

This is pretty amazing: The values of f on the boundary of C completely determine f on the interior of C! This is probably unlike anything you have seen for functions of a real variable.

We will require a mild extension of Cauchy's theorem first.

Lemma 22.5. *Suppose that D is a simply connected region, and let $z_0 \in A$. Suppose g is holomorphic on $A - \{z_0\}$ and continuous on A. Then for all closed curves $C \subset A$, we have*

$$\int_C g dz = 0.$$

Proof. We can deform C to a circle of sufficiently small radius r centered at z_0:

$$\Big| \int_C g dz \Big| = \Big| \int_{|z-z_0|=r} g dz \Big| \leq 2\pi r \max_{|z-z_0|=r} |g(z)|.$$

The theorem follows from taking $r \to 0$. □

Proof of Cauchy Integral Formula. Let A be the interior of C. Define

$$g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$

Since g is continuous on A and differentiable on $A - \{z_0\}$, we have

$$0 = \int_C g dz = \int_C \frac{f(z)-f(z_0)}{z-z_0} dz = \int_C \frac{f(z)}{z-z_0} dz - \int_C \frac{f(z_0)}{z-z_0} dz = \int_C \frac{f(z)}{z-z_0} dz - f(z_0) 2\pi i.$$

□

Theorem 22.6 (Cauchy Integral Formula for derivatives)**.** *Let $n \geq 0$ be an integer. Under the same hypotheses as before,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \qquad 0! = 1, \quad f^{(0)} = f.$$

Proof. Note that

$$f(z_0 + h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0 - h} dz, \qquad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz.$$

With a little algebraic manipulation, we have

$$\begin{aligned} \Big| \frac{f(z+h) - f(z)}{h} - \int_C \frac{f(z)}{(z-z_0)^2} dz \Big| &= \Big| \frac{1}{2\pi i} \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)} dz \Big| \\ &= c_{C,f} |h| \end{aligned}$$

for some constant $c_{C,f} > 0$ independent of h. Now take $|h| \to 0$ to handle $n = 1$. We can proceed inductively for $n \geq 2$. □

23. February 26

Example 23.1. The function

$$g(x) = \begin{cases} (4-x^2)/2 & \text{if } -2 \le x \le 0, \\ (4+x^2)/2 & \text{if } 0 < x \le 2 \end{cases}$$

as a function of a **real variable** is differentiable at $x = 0$, but it is not twice differentiable (the derivative is $|x|$).

Theorem 23.2. *If f is holomorphic at $z_0 \in \mathbb{C}$, then its derivatives of all orders are holomorphic at z_0.*

Proof. Suppose that f is holomorphic at z_0. Then there must exist some $\varepsilon > 0$ such that f is also holomorphic on the disc $D = \{z \in \mathbb{C} \colon |z - z_0| < \varepsilon\}$. We then have

$$f''(z) = \frac{2!}{2\pi i} \int_{|z-w|=\varepsilon/2} \frac{f(w)}{(w-z)^3} dz$$

at all points in the disc $\{z \colon |z - z_0| < \varepsilon/2\}$. Thus $f'(z)$ is holomorphic at z_0 (since f'' exists in a neighborhood of z_0). Similarly, one can use the holomorphy of f' to ensure the holomorphy of f''', and then proceed inductively. □

Corollary 23.3. *If $f = u + iv$ is holomorphic at $z_0 = (x, y)$, then u and v have continuous partial derivatives of all orders at (x, y).*

Corollary 23.4 (Morera)**.** *Let f be continuous on D. If $\int_\gamma f dz = 0$ for every closed contour γ in D, then f is holomorphic on D. (partial converse to Cauchy–Goursat)*

Proof. We showed earlier under these hypotheses that f has a holomorphic anti-derivative F on D. Thus $f = F'$. Since F is holomorphic, so is f by the previous theorem. □

Theorem 23.5. *Suppose that f is holomorphic inside an on a positively oriented circle C_R of radius R centered at z_0. Let M_R be the maximum value of $|f(z)|$ on C_R. then*

$$|f^{(n)}(z_0)| \le n! M_R / R^n, \qquad n = 1, 2, 3, \ldots$$

Proof. Use the triangle inequality for integrals on the Cauchy integral formula. □

24. February 28

Recall that a function is **entire** if it is holomorphic on all of $\mathbb{C}$.

Theorem 24.1 (Liouville). *A bounded entire function must be constant.*

Proof. If f is a bounded entire function, then there exists a constant $M > 0$ such that for any $z_0 \in \mathbb{C}$ and on any circle C_R centered at z_0, we have $|f(z)| \leq M$ for all z on C_R. Thus

$$|f'(z_0)| \leq M/R.$$

But R can be made arbitrarily large, and z_0 was arbitrary. Thus $|f'(z_0)| = 0$ for all $z_0 \in \mathbb{C}$. Hence $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$, and f is constant. ☐

Corollary 24.2 (The fundamental theorem of algebra). *Let $n \geq 1$, and let $a_0, a_1, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$. There exists $z_0 \in \mathbb{C}$ such that the polynomial*

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

satisfies $P(z_0) = 0$.

Proof. We proceed by contradiction. Suppose to the contrary that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P(z)$ is entire. Note that

$$\frac{1}{P(z)} = \frac{1}{z^n} \cdot \frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n}.$$

As $|z| \to \infty$, the sum in the denominator tends to $a_n \neq 0$, and $1/z^n \to 0$. Thus

$$\lim_{|z| \to \infty} \frac{1}{P(z)} = 0.$$

Hence there exists $R > 0$ such that $|1/P| \leq 1$ outside of the circle $|z| = R$. Inside the circle $|z| = R$, $1/P$ is continuous, hence bounded. Thus $1/P$ is bounded everywhere. By Liouville's theorem, $1/P$ is constant. Hence P is constant. But the only constant polynomial has degree $n = 0$, and we assumed that $n \geq 1$, a contradiction. Thus P must have a zero somewhere on $\mathbb{C}$. ☐

Corollary 24.3. *If P is a polynomial of degree n with coefficients in $\mathbb{C}$ (leading coefficient nonzero), then P_n has n complex zeros (not necessarily distinct).*

Proof. We proceed inductively. For $n = 1$, this is clear. Suppose that degree n polynomials have n complex roots. Then a degree $n+1$ polynomial has a least one root by the fundamental theorem, say z_0 and factors into $(z - z_0)Q(z)$, where Q has degree n. Thus the result follows by mathematical induction. ☐

The Cauchy Integral Formula tells us that information about a holomorphic f on the boundary of a region informs us about f on the entire region. We now present an important consequence of this fact. We first require a lemma from calculus, whose proof is easy and we omit.

Theorem 24.4 (Maximum modulus principle). *If f is holomorphic and non-constant on a region U, then $|f(z)|$ achieves its maximum on ∂U.*

Proof. We will prove the result when U is a disc. The general result builds on this case, but the general result is tedious. See the book for the whole proof.

Let $z_0 \in U - \partial U$. Suppose that $|f(z)| \leq |f(z_0)|$ in some NBHD $\{z\colon |z - z_0| < \varepsilon\}$. Pick a circle γ of radius $0 < r \leq \varepsilon$ centered at z_0. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Note that

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| d\theta = |f(z_0)|,$$

so

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta.$$

Hence

$$\int_0^{2\pi} (|f(z_0)| - |f(z_0 + re^{i\theta})|) d\theta = 0.$$

Since the integrand is a continuous function of θ and the integral is zero, the integrand must be zero. Thus

$$|f(z_0)| = |f(z_0 + re^{i\theta})|, \qquad \theta \in [0, 2\pi].$$

But this holds for all $0 < r \leq \varepsilon$. So everywhere in the ε-NBHD, we have $|f(z)| = |f(z_0)|$. By Midterm Exam Problem 4, it follows that $f(z) = f(z_0)$ everywhere in the ε-NBHD. □

Corollary 24.5. *If u is non-constant and harmonic on a region U, then $|u|$ achieves its maximum on ∂U.*

Proof. There exists a function f, holomorphic on U, such that $\text{Re} f = u$. Then e^f is holomorphic on U and non-constant on U. Thus $|e^f| = e^u$ achieves its maximum on ∂U. □

25. March 9-11

We will now switch gears and start discussing power series.

Definition 25.1. *A sequence of complex numbers $(z_n)_{n\geq 1}$ has a limit z if, for each $\varepsilon > 0$, there exists an integer $N > 0$ such that $|z_n - z| < \varepsilon$ whenever $n > N$. When the limit exists, we say that (z_n) converges to z, and we write*

$$\lim_{n\to\infty} z_n = z, \qquad z_n \to z.$$

Otherwise, (z_n) diverges.

Theorem 25.2. *We have $z_n \to z$ if and only if $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$.*

Definition 25.3. *Given a sequence (z_n), we define the sequence of partial sums (S_n) by*

$$S_n = \sum_{k=1}^{n} a_n.$$

If (S_n) converges to a limit S, then the infinite series $\sum_{k=1}^{\infty} z_k$ converges to S. Otherwise, the infinite series diverges.

Theorem 25.4. *We have $\sum_{n=1}^{\infty} z_n = L$ if and only if*

$$\sum_{n=1}^{\infty} \mathrm{Re}(z_n) = \mathrm{Re}(L), \qquad \sum_{n=1}^{\infty} \mathrm{Im}(z_n) = \mathrm{Im}(L).$$

Corollary 25.5. *If $\sum z_n$ converges, then $z_n \to 0$.*

Exercise: Convergent sequences are **bounded** (i.e. there exists a constant $M \geq 0$ such that $|z_n| \leq M$ for all n).

Definition 25.6. *A series $\sum z_n$ converges absolutely if $\sum |z_n|$ converges.*

Corollary 25.7. *Absolute convergence of an infinite series implies convergence of that infinite series.*

Given a sequence (z_n) and the sequence of partial sums (S_n), if $S_n \to S$, we define the **remainder**

$$\rho_n = S - S_n.$$

We have that $S_n \to S$ if and only if $\rho_n \to 0$. This will be important in our study of **power series**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where z_0 and a_n are complex constants, and z lies in a given region containing z_0. In such series, involving a variable z, we denote sums, partial sums, and remainders by $S(z), S_n(z), \rho_n(z)$.

Definition 25.8. *A function f on an open set $D \subset \mathbb{C}$ is* **analytic at** *$z_0 \in D$ if f is infinitely differentiable at z_0 with Taylor series*

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converging pointwise to $f(z)$ for all z in a neighborhood of z_0. Moreover, f is **analytic on** *D if it is analytic at each $z_0 \in D$.*

There is a version of this definition for real-valued functions too. For instance,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

are real-analytic on all of $\mathbb{C}$.

Theorem 25.9. *A function f is holomorphic on D if and only if it is analytic on D.*

Example 25.10. Let

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

All derivatives of f exist at $x = 0$ (an exercise in l'Hospital's rule). In fact, $f^{(n)}(0) = 0$ for all $n \geq 0$. But f is not identically zero in a neighborhood of zero, so f is not analytic at zero, even though all of its derivatives exist there (and everywhere else).

Proof. We will prove that holomorphy implies analyticity. The converse direction is immediate once we justify out ability to differentiate power series term by term when in the region of absolute convergence, which we will discuss later.

Since holomorphy is defined in terms of discs, it suffices to check this on discs centered at $z_0 \in D$. By a change of variables, it suffices for us to consider f on and inside a circle C centered at $z = 0$ of radius $R_0 > 0$. Let $0 < r < r_0 < R_0$. Let C be the boundary of $\{|z| \leq R_0\}$, and let $|z| = r$. If f is holomorphic on C_0, then

$$\begin{aligned} f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(w)}{w-z}dw &= \frac{1}{2\pi i}\int_C \frac{1}{w}\cdot\frac{1}{1-\frac{z}{w}}f(w)dw \\ &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w}\Big(\sum_{n=0}^{\infty}\Big(\frac{z}{w}\Big)^n\Big)dw. \end{aligned}$$

Since $0 < r < r_0 < R_0$, we have $|z/w| < 1$. Thus by properties of geometric series, the above equals, for any $N \geq 2$,

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w}\Big(\Big(\sum_{n=0}^{N}\frac{z}{w}\Big)^n + \frac{(z/w)^N}{1-z/w}\Big)dw = \sum_{n=0}^{N-1}\frac{z^n}{n!}\Big(\frac{n!}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}dw\Big) + z^N\int_C \frac{f(w)}{(w-z)w^N}dw.$$

Since $|z| = r < r_0 < R_0 = |w|$, we have by the triangle inequality for integrals that

$$\Big|z^N\int_C \frac{f(w)}{(w-z)w^N}dw\Big| \leq \Big(\frac{r}{R_0}\Big)^N \frac{M_{R_0}}{1-\frac{r}{R_0}}\cdot 2\pi R_0.$$

Thus by the Cauchy Integral Formula,

$$0 \leq \lim_{N\to\infty}\Big|f(z) - \sum_{n=0}^{N-1}\frac{f^{(n)}(0)}{n!}z^n\Big| = \lim_{N\to\infty}\Big|z^N\int_C \frac{f(w)}{(w-z)w^N}dw\Big| \leq \frac{2\pi M_{R_0}}{R_0 - r}\lim_{N\to\infty}\Big(\frac{r}{R_0}\Big)^N = 0.$$

□

26. March 13

We are now in a position to justify our identity $e^{i\theta} = \cos\theta + i\sin\theta$. One of several equivalent definitions of e^x (as a function of a real variable) is the solution to the differential equation

$$f'(x) = f(x), \qquad f(0) = 1.$$

We now consider the same differential equation

$$f'(z) = f(z), \qquad f(0) = 1.$$

Any solution to this must be infinitely differentiable ($f(z) = f'(z) = f'(z) = f''(z) = \cdots$) at $z = 0$, so let us assume a Taylor series of the shape (note $a(0) = 1$ by hypothesis)

$$f(z) = \sum_{n=0}^{\infty} \frac{a(n)}{n!} z^n = 1 + \sum_{n=1}^{\infty} \frac{a(n)}{n!} z^n = 1 + \frac{a(1)}{1!} z + \frac{a(2)}{2!} z^2 + \frac{a(3)}{3!} z^3 + \cdots$$

We now observe that

$$f'(z) = \sum_{n=1}^{\infty} \frac{a(n)}{(n-1)!} z^{n-1} = a(1) + \frac{a(2)}{1!} z + \frac{a(3)}{2!} z^2 + \frac{a(4)}{3!} z^3 + \cdots.$$

By comparing coefficients, we have $1 = a(1)$ and $a(n) = a(n+1)$ for all $n \geq 1$. Thus

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is the solution. Since this is precisely the Taylor series of e^x with x replaced by z, we are then justified in defining

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

Thus $e^{i\theta}$ is now defined in terms of an absolutely convergent power series. Similarly, we define

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

and we recover the more general identity

$$e^{iz} = \cos z + i \sin z.$$

Section 64 contains several other examples of power series expansions for various functions, and shows how to use construct power series for new functions building off of power series known for old functions.

But what if f is **not** analytic at a point z_0? Is it still possible to talk about a power series expansion at $z = z_0$?

Example 26.1. The function e^z/z is not holomorphic (or even defined) at $z = 0$. However, if $|z| > 0$, then we may freely write

$$\frac{e^z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}.$$

This equals $\frac{1}{z}$ plus the Taylor series centered at zero for

$$g(z) = \begin{cases} (e^z - 1)/z & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

(Check that g is holomorphic.) More generally, for an integer $m \geq 1$,

$$\frac{e^z}{z^m} = \frac{1}{z^m}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^m}\sum_{n=0}^{m-1}\frac{z^n}{n!} + \frac{1}{z^m}\sum_{n=m}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{m-1}\frac{1}{z^{m-n}n!} + \sum_{n=m}^{\infty}\frac{z^{n-m}}{n!}.$$

(The infinite sum is the Taylor series of which function?)

Example 26.2. We have

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}, \qquad z \in \mathbb{C}.$$

Hence if $z \neq 0$, then

$$\cos(1/z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)! z^{2n}}$$

Example 26.3. Let

$$\frac{1+z^2}{1-z^2}, \qquad z \neq \pm 1.$$

Note that $|z| < 1$ if and only if $|z|^2 < 1$. Hence we can use

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1$$

to deduce

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}, \qquad |z| < 1.$$

Moreover,

$$z^2\sum_{n=0}^{\infty} z^{2n} = \sum_{n=0}^{\infty} z^{2n+2}, \qquad |z| < 1.$$

Hence

$$\frac{1+z^2}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} + \sum_{n=0}^{\infty} z^{2n+2} = 1 + 2\sum_{n=1}^{\infty} z^{2n}, \qquad |z| < 1.$$

NOTE: We can only perform this sort of addition in the region of absolute convergence.

27. March 16

If f is analytic at $z = z_0$, then f has a Taylor expansion in some disc centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad |z - z_0| < R.$$

There is a more general notion of a power series, one that incorporates negative powers of $z - z_0$. The function $f(z)$ has a **Laurent series** centered at z_0 if

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n, \qquad R_1 < |z - z_0| < R_2.$$

Note that the region

$$\{z\colon R_1 < |z - z_0| < R_2\}$$

is an **annulus** (like a donut or a washer).

We need R_1 to give us a healthy separation from z_0 because we could be summing infinitely many negative powers of $z - z_0$, depending on the coefficients.

Theorem 27.1 (Laurent's theorem)**.** *Suppose that f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C be a positively oriented simple closed curve around z_0 lying in the annulus. Then at each point in the domain,*

$$f(z) = \sum_{n\in\mathbb{Z}}^{\infty} a_n(z - z_0)^n, \qquad R_1 < |z - z_0| < R_2,$$

where for each $n \in \mathbb{Z}$, we have

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Note that if f is actually analytic in the disc $|z - z_0| < R_2$, then for $n \geq 1$,

$$a_{-n} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz = \frac{1}{2\pi i} \int_C f(z)(z - z_0)^{n-1} dz = 0$$

and for $n \geq 0$ by Cauchy–Goursat. Thus

$$a_n = \begin{cases} f^{(n)}(z_0)/n! & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Thus we recover Taylor's theorem.

Proof. This proof is a bit different from the book's proof. It relies on the fact that we have already proved Taylor's theorem.

It suffices to show that f can be written as $f_1 + f_2$, where f_1 is analytic for $|z - a| < R_2$ and f_2 is analytic for $|z - a| > R_1$, where $\lim_{z\to\infty} f(z)$ is *removable* (more on this later). Under these circumstances, f_1 can be written as a power series in $z - z_0$, and f_2 a power series in $(z - z_0)^{-1}$.

To find the representation $f = f_1 + f_2$, we define

$$f_1(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r_1} \frac{f(w)}{w - z} dw, \qquad |z - z_0| < r_1 < R_2,$$

and

$$f_2(z) = -\frac{1}{2\pi i}\int_{|w-z_0|=r_2} \frac{f(w)}{w-z}dw, \qquad R_1 < r_2 < |z-z_0|.$$

By the principle of deformation, the values of r_1, r_2 are irrelevant as long as

$$R_1 < r_2 < |z-z_0| < r_1 < R_2.$$

For this reason, f_1 and f_2 are uniquely defined and represent analytic functions in $|z-z_0| < R_2$ and $|z-a| > R_1$, respectively. Moreover, by the Cauchy Integral Formula, $f = f_1 + f_2$.

The Taylor series for f_1 is

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad a_n = \frac{1}{2\pi i}\int_{|w-z_0|=r_1} \frac{f(w)}{(w-z_0)^{n+1}}dw.$$

To find the series for f_2, we perform the change of variables

$$w = a + 1/w', \qquad z = z_0 + 1/z'.$$

This transforms the circle $|w-z_0| = r_2$ (with positive orientation) to $|w'| = 1/r_2$ (with negative orientation), and we find that

$$f_2\Big(z_0 + \frac{1}{z'}\Big) = \frac{1}{2\pi i}\int_{|w'|=\frac{1}{r_2}} \frac{z'}{w'}\frac{f(z_0+\frac{1}{w'})}{w'-z'}dw'.$$

We now proceed as in our proof of Taylor's theorem and deduce that

$$f_2\Big(z_0 + \frac{1}{z'}\Big) = \sum_{n=1}^{\infty} b_n(z')^n,$$

where

$$b_n = \frac{1}{2\pi i}\int_{|w'|=\frac{1}{r_2}} \frac{f(z_0+\frac{1}{w'})}{(w')^{n+1}}dw' = \frac{1}{2\pi i}\int_{|w-a|=r_2} f(w)(w-z_0)^{n-1}dw.$$

The desired result now follows from unraveling our changes of variables. □

Example 27.2. Let

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z}\cdot\frac{1}{1+z^2}.$$

This has singularities at $z = 0$ and $z = \pm i$. We will find the Laurent expansion of f in the region $0 < |z| < 1$. In this region, we have

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}, \qquad |z| < 1.$$

Thus

$$\frac{1}{z(z^2+1)} = \sum_{n=0}^{\infty}(-1)^n z^{2n-1} = \sum_{n=-1}^{\infty}(-1)^{n+1}z^{2n+1} = \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}, \qquad 0 < |z| < 1.$$

28. March 18

We want to be able to perform limit operations (integrals, derivatives, usual limits, etc.) to power series by performing the limit operations (integrals, derivatives, usual limits, etc.) on the individual terms. We now build up toward justifying when we can do this.

Recall that a series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem 28.1. *If a power series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges when $z = z_1 \neq z_0$, then it converges absolutely for each z in the open disc $|z - z_0| < R_1 := |z_1 - z_0|$.

Proof. If $z_1 \neq z_0$ and

$$\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$

converges, then the sequence $a_n(z_1 - z_0)^n$ is bounded; that is, there exists a constant $M > 0$ such that

$$|a_n(z_1 - z_0)^n| \leq M, \qquad n \geq 0.$$

If $|z - z_0| < R_1 = |z_1 - z_0|$ and we write

$$\rho = \frac{|z - z_0|}{|z_1 - z_0|},$$

then

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \Big(\frac{|z - z_0|}{|z_1 - z_0|} \Big)^n \leq M\rho^n, \qquad n \geq 0,$$

and

$$\sum M\rho^n$$

is a convergent geometric sum. Thus the power series converges absolutely by the comparison test. □

The greatest circle centered at z_0 such that the power series converges at each point inside is called the **circle of convergence**. The power series cannot converge outside of the circle, but it might converge (perhaps conditionally) on the boundary of the circle.

Our next definition involves some new terminology. Suppose our power series has circle of convergence $|z - z_0| = R$, and let

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad S_N(z) = \sum_{n=0}^{N-1} a_n (z - z_0)^n \quad |z - z_0| < R.$$

Define

$$\rho_N(z) = S(z) - S_N(z), \qquad |z - z_0| < R.$$

Inside of the circle of convergence, we know that $\lim_{N\to\infty} \rho_N(z) = 0$. In other words, for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|\rho_N(z)| < \varepsilon \qquad \text{whenever} \quad n \geq N.$$

Definition 28.2. *When the choice of N depends* **only** *on ε and is independent of the point z taken in the specified region within the circle of convergence, the convergence of $S_N(z)$ to $S(z)$ is said to be* **uniform**.

Theorem 28.3. *If z_1 is a point in the interior the circle of convergence $|z - z_0| = R$ of a power series $\sum a_n(z - z_0)^n$, then the series converges uniformly in the closed disc $|z - z_0| \leq R_1 := |z_1 - z_0|$.*

29. March 20

Theorem 29.1. *If z_1 is a point in the interior the circle of convergence $|z-z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, then the series converges uniformly in the closed disc $|z-z_0| \leq R_1 := |z_1 - z_0|$.*

A similar proof holds for Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in a suitable annular domain $R_1 < |z-z_0| < R_2$, but it is very similar to what we just did for Taylor series. So I will just work the proof for Taylor series.

Proof. At our point z_1, we are in the circle of absolute convergence. Thus

$$\sum_{n=0}^{\infty} |a_n(z_1-z_0)^n|$$

converges. We then have

$$\rho_N(z) = S(z) - S_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n(z-z_0)^n.$$

Define

$$\sigma_N = \lim_{m\to\infty} \sum_{n=N}^{m} |a_n(z_1-z_0)^n|.$$

If $|z-z_0| \leq R_1 = |z_1 - z_0|$, then it follows from Ex.3, Sec.61 and the triangle inequality that

$$|\rho_N| \leq \sigma_N, \qquad |z-z_0| \leq R_1.$$

Since $\lim_{N\to\infty} \sigma_N = 0$ (we are in the region of absolute convergence), it follows that for all $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $\sigma_N < \varepsilon$ when $N > N_\varepsilon$. Since this holds for all points $|z-z_0| \leq R_1$, the value of N_ε is independent of z. So the convergence is uniform. □

Theorem 29.2. *A power series $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ represents a continuous function at each point in its circle of convergence $|z-z_0| = R$.*

A similar proof holds for Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in a suitable annular domain $R_1 < |z-z_0| < R_2$, but it is very similar to what we just did for Taylor series. So I will just work the proof for Taylor series.

Proof. Let z, z_1 be in the circle of convergence. Our goal is to prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(z) - S(z_1)| < \varepsilon, \qquad |z - z_1| < \delta.$$

So, we let $\varepsilon > 0$. For $|z-z_0| < R$, we have $S(z) = S_N(z) + \rho_N(z)$. Now, if z_1 lies in the circle of convergence, then

$$|S(z) - S(z_1)| = |S_N(z) + \rho_N(z) - S_N(z_1) - \rho_N(z_1)| \leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)|.$$

By uniform convergence, we know that there exists $N_\varepsilon > 0$ such that

$$|\rho_N(z)|, |\rho_N(z_1)| < \frac{\varepsilon}{3}, \qquad N > N_\varepsilon.$$

Since $S_N(z)$ is continuous (it's a polynomial), we have that if $N = N_\varepsilon + 1$, then there exists a sufficiently small $\delta > 0$ such that

$$|S_N(z) - S_N(z_1)| < \frac{\varepsilon}{3}, \qquad |z - z_1| < \delta.$$

Thus $|S(z) - S(z_1)| < \varepsilon$, as desired. □

Theorem 29.3. *Let γ be a contour interior to the circle of convergence of $S(z)$, and let $g(z)$ be continuous on γ. We have*

$$\int_\gamma g(z)S(z)dz = \sum a_n \int_\gamma g(z)(z-z_0)^n dz.$$

Proof. One uses uniform continuity much like the proof of continuity of power series, along with Morera's theorem. See the book. □

For z in a suitable annulus (away from points of discontinuity), the proof also works for Laurent series: The idea works for negative integers as well.

Corollary 29.4. *$S(z)$ is holomorphic at each point z interior to its circle of convergence. If $S(z)$ is a Laurent series, then the same holds in the interior of the annulus of definition.*

Proof. Take $g(z) = 1$ everywhere, in which case the theorem yields $\int_\gamma S(z)dz = 0$. The result now follows by Morera's theorem. □

Corollary 29.5. *In its circle of convergence, $S(z)$ satisfies $S'(z) = \sum_{n=0}^\infty a_n \frac{d}{dz}(z-z_0)^n$. If $S(z)$ is a Laurent series, then the same holds in the interior of the annulus of definition.*

Proof. Take Theorem 29.3 with

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(s-z)^2}$$

and apply the Cauchy Integral Formula for derivatives. □

Theorem 29.6. *If $\sum_{n=0}^\infty a_n(z-z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z-z_0| = R$, then it is Taylor series expansion for f in powers of $z - z_0$.*

If $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ converges to $f(z)$ at all points in some annular domain about z_0, then it is the Laurent series expansion for f in powers of $z-z_0$ in that domain.

Proof. We describe this for Laurent series. We have

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n.$$

Then by the theorem,

$$\int_\gamma g(z)f(z)dz = \sum_{m\in\mathbb{Z}} c_m \int_\gamma g(z)(z-z_0)^m dz.$$

We choose

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}}, \qquad n \in \mathbb{Z}$$

and conclude that

$$c_n = \int_\gamma g(z)f(z)dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

which is precisely the coefficient at n for the Laurent series of f about z_0. □

30. March 23

Cauchy–Goursat: If f is analytic at all points interior to and on a simple closed contour γ, then

$$\int_\gamma f(z)dz = 0.$$

But what if f is not analytic inside γ at, say, finitely many points?

Sec. 25: If f is analytic at z_0, then f is analytic in a NBHD of z_0. If f is not analytic at z_0, but it is analytic at some point in every NBHD of z_0, then z_0 is a singular point of f.

We will begin studying **isolated** singular points, that is, singular points z_0 such that a function f is analytic at $0 < |z - z_0| < \varepsilon$ but not at z_0.

Example 30.1. The isolated singularities of $z/\cos z$ are at $z = \frac{\pi}{2} + \pi n$ for each $n \in \mathbb{Z}$. If p, q are polynomials, then the isolated singularities of $p(z)/q(z)$ are the zeros of $q(z)$.

Example 30.2. The principal branch $\mathrm{Log}(z)$ has a singular point at $z = 0$, but it is not isolated since every NBHD of $z = 0$ intersects the branch cut (negative real axis).

Example 30.3. The isolated singularities of $1/\sin(\pi/z)$ are $z = 1/n$, $n = 1, 2, 3, \ldots$. There is a singularity at $z = 0$, but it is not isolated.

When z_0 is an isolated singular point of f, there is a radius $R_2 > 0$ such that f is analytic on $0 < |z - z_0| < R_2$. Thus f has a Laurent series expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n, \qquad 0 < |z - z_0| < R_2.$$

Let C be a positively oriented simple closed contour around z_0 that lies in the punctured disc $0 < |z - z_0| < R_2$. We have the equality

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}, \qquad n \in \mathbb{Z}.$$

Note that for $n = -1$, we have

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz.$$

This number, the coefficient of $(z - z_0)^{-1}$, is called the **residue** of f at z_0. We write this as

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_C f(z)dz.$$

Example 30.4. Consider

$$\int_{|z|=1} \frac{e^z - 1}{z^4} dz.$$

This is analytic on $\mathbb{C}$ apart from $z = 0$. The Laurent series for the integrand about $z = 0$ is

$$\frac{1}{z^4}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \frac{1}{24} + \frac{z}{120} + \cdots$$

Since the residue at $z = 0$ is $1/6$, the integral equals $2\pi i \cdot 1/6 = \pi i/3$.

Example 30.5.

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

The residue at $z = 0$ is zero, so

$$\int_{|z|=1} \cosh(1/z)dz = 0.$$

Example 30.6.

$$\int_{|z-2|=1} \frac{1}{z(z-2)^5}dz$$

Only singularity inside the contour is at $z = 2$. So we develop the Laurent series about $z = 2$:

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\frac{1/2}{1-\frac{-(z-2)}{2}} = \frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-5}, \qquad 0 < |z-2| < 2.$$

At $n = 4$, the coefficient of $(z-2)^{-1}$, is $1/32$, which is the residue of our integrand at $z = 2$. Hence

$$\int_{|z-2|=1} \frac{1}{z(z-2)^5}dz = 2\pi i(1/32) = \frac{\pi i}{16}.$$

Theorem 30.7 (Cauchy's residue theorem)**.** *Let γ be a (positive) simple closed contour. If f is analytic inside and on γ except for a finite number of singular points $z_1, \ldots, z_n$ interior to γ, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Proof. Choose circles C_k centered at z_k $(1 \le k \le n)$ with radius small enough so that each is contained in γ and no two intersect. (The exact value of the radius won't matter because of the principle of deformation of curves.) Taking the circles C_k in the positive direction, we find that the interior of γ intersected with the exterior of the circles C_k is a multiply connected region on which f is analytic. Therefore, by Cauchy–Goursat for multiply connected domains (Theorem 22.1),

$$\int_\gamma f(z)dz + \sum_{k=1}^{n}\int_{-C_k} f(z)dz = 0.$$

Since $-\int_{-C_k} = \int_{C_k}$, it follows that

$$\int_\gamma f(z)dz = \sum_{k=1}^{n}\int_{C_k} f(z)dz.$$

Since

$$\int_{C_k} f(z)dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z),$$

the proof is finished. □

31. March 25

Example 31.1.

$$\int_{|z|=2} \frac{4z-5}{z(z-1)}dz.$$

The contour contains the two isolated singularities of f at $z=0$ and $z=1$. Thus

$$\int_{|z|=2} \frac{4z-5}{z(z-1)}dz = 2\pi i\Big(\operatorname*{Res}_{z=0} \frac{4z-5}{z(z-1)} + \operatorname*{Res}_{z=1} \frac{4z-5}{z(z-1)}\Big).$$

For the singularity at $z=0$, we note that

$$\frac{4z-5}{z(z-1)} = -\frac{4z-5}{z}\frac{1}{1-z} = \Big(\frac{5}{z}-4\Big)\sum_{n=0}^{\infty} z^n = \frac{5}{z}\sum_{j=0}^{\infty} z^j - 4\sum_{j=0}^{\infty} z^n, \qquad |z|<1.$$

Thus

$$\operatorname*{Res}_{z=0} \frac{4z-5}{z(z-1)} = 5.$$

For the singularity at $z=1$, we note that

$$\frac{4z-5}{z(z-1)} = \frac{4(z-1)-1}{z-1}\frac{1}{1+(z-1)} = \Big(4-\frac{1}{z-1}\Big)\sum_{k=0}^{\infty}(-(z-1))^k, \qquad 0<|z-1|<1.$$

A similar calculation shows that

$$\operatorname*{Res}_{z=1} \frac{4z-5}{z(z-1)} = -1.$$

Thus the full integral is

$$2\pi i(5+(-1)) = 8\pi i.$$

For both residues, one could alternatively use the partial fraction decomposition

$$\frac{4z-5}{z(z-1)} = \frac{5}{z} - \frac{1}{z-1},$$

but such a nice decomposition is never guaranteed.

Suppose f is analytic on $\mathbb{C}$ except for finitely many singular points. There exists some circle C with sufficiently large radius that encompasses all of the the singular points. The residue theorem tells us how to evaluate

$$\int_C f(z)dz,$$

where C is positively oriented. But if C is being *negatively oriented* (denoted $-C$), then the isolated singularities do not lie interior to C any longer. However, the point at infinity does. (You can think of this as on the Riemann sphere.) In this situation, it is appropriate to think of infinity as an isolated singularity of f (though it does not lie in $\mathbb{C}$). Now, by our principle of deformation,

$$-\int_C f(z)dz = \int_{-C} f(z)dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

To find the residue, we consider the Laurent expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n, \qquad c_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{z^{n+1}}, \qquad R_1 < |z| < \infty.$$

We evaluate f at $1/z$ and multiply the ensuing sum by $1/z^2$, obtaining

$$\frac{1}{z^2}f(1/z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n+2}}, \qquad 0 < |z| < R_1^{-1}.$$

Thus $\operatorname{Res}_{z=\infty} f(z) = \operatorname{Res}_{z=0} \frac{1}{z^2} f(1/z)$. Sometimes, but not always, this is easier to compute. It depends on the function f, the particular singularities, what contour C is, etc.

Let f have an isolated singularity at $z = z_0$. Consider the Laurent series

$$f(z) = T(z) + P(z), \qquad 0 < |z - z_0| < R_2.$$

where

$$T(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is a Taylor series and

$$P(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is the **principal part**.

The principal part might have zero terms, in which case we call z_0 a **removable singularity** of f. The principal part might have m terms (with $1 \le m < \infty$), in which case we call z_0 a **pole of order** m of f. The principal part might have infinitely many terms, in which case we call z_0 an **essential singularity** of f.

Example 31.2. Let

$$f(z) = \frac{e^z - 1}{z}, \qquad z_0 = 0.$$

Then in the annular region $0 < |z| < \infty$, we have

$$f(z) = \frac{1}{z}\Big[-1 + \Big(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\Big)\Big] = \frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Thus f has a removable singularity with residue 0 at $z = 0$. Moreover, $\lim_{z\to 0} f(z) = 1$, and f can be extended to a function which is holomorphic at $z = 0$. In particular, the function $\widetilde{f}(z)$ given by 0 when $z = 0$ and $(e^z - 1)/z$ when $z \neq 0$ is entire.

Example 31.3. For $0 < |z| < \infty$, we have

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots$$

Thus the principal part is infinite, and f has an essential singularity with residue 1 at $z = 0$.

Theorem 31.4 (Picard)**.** *In each NBHD of an essential singularity, a function assumes every finite value, with one possible exception, an infinite number of times.*

Example 31.5. For $f(z) = e^{1/z}$, we have $e^{1/z} = -1$ an infinite number of times ($z = 1/((2n+1)\pi i)$, $n \in \mathbb{Z}$). Also, f is not defined at $z = 0$, but it is holomorphic elsewhere.

Example 31.6. For $0 < |z| < \infty$, we have

$$f(z) = \frac{e^z}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.$$

Thus f has a pole of order 1 with residue 1 at $z = 0$.

32. March 27

We now introduce an analytic expression for the residue of f at a pole $z = z_0$ of finite order $m \geq 1$. For $0 < |z - z_0| < R_2$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-m}}{(z-z_0)^m}.$$

Multiply both sides by $(z - z_0)^m$:

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + a_{-1}(z-z_0)^{m-1} + a_{-2}(z-z_0)^{m-2} + \cdots + a_{-m}.$$

Take the $(m-1)$-th derivative of both sides:

$$\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z) = \sum_{n=0}^{\infty}(n+m)(n+m-1)\cdots(n+2)a_n(z-z_0)^{n+1} + (m-1)! a_{-1}.$$

Take the limit of both sides as $z \to z_0$:

$$\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z) = a_{-1}(m-1)!.$$

Now, divide both sides by $(m-1)!$ to obtain

$$\frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z) = a_{-1} = \operatorname*{Res}_{z=z_0} f(z).$$

This can be extended to $m = 0$ if we define the d/dz^{-1} to be identity function.

This calculation also shows:

Theorem 32.1. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of order $m \geq 1$ of f if and only if there exists a function $\phi(z)$, holomorphic and nonzero in a NBHD of z_0, such that*

$$f(z) = \phi(z)(z-z_0)^{-m}, \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

We now prove a convenient result in the special case:

Lemma 32.2. *Suppose that f has an isolated singularity at z_0, and in a neighborhood of z_0, there exist holomorphic functions g and h such that*

$$f(z) = \frac{g(z)}{h(z)},$$

where $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}.$$

Proof. By l'Hopital's rule,

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{zg(z) - z_0 g(z)}{h(z)} = \lim_{z\to z_0}\frac{g(z) + zg'(z) - z_0 g'(z)}{h'(z)} = \frac{g(z_0)}{h'(z_0)}.$$

□

Example 32.3. We have already proved that $\text{Res}_{z=0} e^z/z = 1$. More generally, if $m \geq 1$, then

$$\operatorname*{Res}_{z=0} \frac{e^z}{z^m} = \frac{1}{(m-1)!} \lim_{z \to 0} \frac{d}{dz^{m-1}} e^z = \frac{1}{(m-1)!}.$$

Example 32.4. Let

$$f(z) = \frac{\sin z}{z(z-1)}.$$

The isolated singularity at $z = 0$ is removable (we calculated this a while ago), and the isolated singularity at $z = 1$ is a pole of order 1 (a **simple pole**). Thus by our calculation above,

$$\operatorname*{Res}_{z=1} \frac{\sin z}{z(z-1)} = \left. \frac{\sin z}{\frac{d}{dz}(z(z-1))} \right|_{z=1} = \sin 1.$$

As we have seen, zeros and poles are closely related. Much like the order of a pole, the **order of a zero** $z = z_0$ is the positive integer m such that

$$f^{(j)}(z_0) = 0, \ 0 \leq j \leq m-1, \qquad f^{(m)}(z_0) \neq 0.$$

Theorem 32.5. *Let f be analytic at z_0. Then f has a zero of order m at z_0 if and only if $f(z) = (z - z_0)^m g(z)$, where g is analytic and nonzero in a NBHD of z_0.*

Proof. Suppose f has a zero of order m at z_0. The Taylor expansion of f about z_0 is then

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n, \qquad a_m \neq 0, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Thus

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^{m+n}.$$

Since $a_m \neq 0$, we may take

$$g(z) = \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^{m+n}.$$

Conversely, if $f(z) = (z - z_0)^m g(z)$ with $g(z)$ holomorphic and nonzero in a NBHD of z_0. Then

$$f(z) = (z - z_0)^m \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^{m+n}.$$

Now,

$$f^{(j)}(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (m+n)(m+n-1) \cdots (m+n-j+1)(z - z_0)^{m+n-j}.$$

If $0 \leq j \leq m-1$, this expression equals zero at $z = z_0$. Since $g(z_0) \neq 0$, this expression is nonzero at $z = z_0$ when $j = m$. $\square$

Example 32.6. $f(z) = z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$ has four zeros of order 1, at each 4th root of unity.

Much like isolated singularities, a function f, analytic at z_0, has an **isolated zero** at z_0 if there is a deleted NBHD $0 < |z - z_0| < \varepsilon$ in which f is nonzero.

Theorem 32.7. *If f is analytic at z_0, and $f(z_0) = 0$ but f is not identically zero in any NBHD of z_0, then $f \neq 0$ in some deleted NBHD of z_0.*

Proof. See Section 82, theorem 2. □

Theorem 32.8. *If f in analytic in a NBHD N_0 of z_0, and $f(z) = 0$ at each z of a domain D or a line segment L containing z_0, then $f \equiv 0$ on N_0.*

Proof. □

33. March 30

Let us recall a lemma from last time that will be useful for many upcoming examples.

Lemma 33.1. *Suppose that f has an isolated singularity at z_0, and in a deleted neighborhood of z_0, there exist holomorphic functions g and h such that*

$$f(z) = \frac{g(z)}{h(z)},$$

where $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}.$$

Proof. By l'Hopital's rule,

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0} \frac{zg(z) - z_0 g(z)}{h(z)} = \lim_{z\to z_0} \frac{g(z) + zg'(z) - z_0 g'(z)}{h'(z)} = \frac{g(z_0)}{h'(z_0)}.$$

□

Lemma 33.2. *Suppose that f has an isolated singularity at z_0, and z_0 is a pole of order 1.*

33.1. **Rational functions of trig functions.** Let

$$R(x,y) = \frac{p(x,y)}{q(x,y)}$$

be a ratio of polynomials in two variables without a pole on the circle $x^2 + y^2 = 1$. Then by the change of variables $z = e^{it}$ (note that $\overline{z} = z^{-1}$ now), we have

$$z = x + iy, \qquad x = \cos t = \frac{z + z^{-1}}{2}, \qquad y = \sin t = \frac{z - z^{-1}}{2i}$$

and

$$\int_0^{2\pi} R(\sin t, \cos t)dt = \int_{|z|=1} R\Big(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\Big)\frac{dz}{iz} = \frac{2\pi i}{i}\sum_{j=1}^{k} \operatorname*{Res}_{z=z_k} \frac{1}{z} R\Big(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\Big),$$

where z_k ranges over the poles of the integrand inside the circle $|z| = 1$.

Example 33.3. Let $a > 1$, and consider the integral

$$\int_0^{2\pi} \frac{dt}{a + \sin t}.$$

Then

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \int_{|z|=1} \frac{1}{a + \frac{z - z^{-1}}{2i}} \frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{2i}{z^2 + 2aiz - 1} dz.$$

The integrand has one pole in the circle, namely

$$z_0 = -ai + i\sqrt{a^2 - 1}.$$

The pole is of order 1, so we apply Lemma 33.1 to compute the residue:

$$\operatorname*{Res}_{z=z_0} \frac{2i}{z^2 + 2aiz - 1} = \frac{2i}{2z + 2ia}\bigg|_{z=z_0} = \frac{i}{z_0 + ai} = \frac{1}{\sqrt{a^2 - 1}},$$

so the integral equals

$$\frac{1}{i}2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{z^2+2aiz-1} = \frac{2\pi}{\sqrt{a^2-1}}.$$

33.2. **Rational functions with no real poles.** Let $r(x) = p(x)/q(x)$ be a rational function with no real poles. Suppose that

$$\lim_{|z|\to\infty} zr(z) = 0.$$

Consider the contour $\gamma = [-R, R] \cup \gamma_R$, where γ_R is parametrized by Re^{it}, $0 < t < \pi$. Then

$$\int_{-R}^{R} r(x)dx + \int_{\gamma_R} r(z)dz = 2\pi i \sum_{Im(z_k)>0} \operatorname*{Res}_{z=z_k} r(z),$$

where z_k ranges over the poles of $r(z)$ lying inside γ. We now take $R \to \infty$. Note that

$$\Big| \int_{\gamma_R} r(z)dz \Big| \le \pi R \cdot \max_{z\in\gamma_R} |f(z)|,$$

which tends to 0 as $R \to \infty$ by hypothesis. Thus

$$\int_{-\infty}^{\infty} r(x)dx = 2\pi i \sum_{Im(z_k)>0} \operatorname*{Res}_{z=z_k} r(z),$$

where z_k runs over *all* poles of $r(z)$ with $\mathrm{Im}(z_k) > 0$. By identical methods, we could also prove

$$\int_{-\infty}^{\infty} r(x)dx = -2\pi i \sum_{Im(z_k)<0} \operatorname*{Res}_{z=z_k} r(z).$$

Example 33.4. Consider

$$\int_0^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1}.$$

Clearly

$$\lim_{|z|\to\infty} \frac{z}{z^6+1} = 0.$$

The integrand has six poles: $z_k = e^{\pi i/6 + \pi i k/3}$, $k = 0, 1, 2, 3, 4, 5$. When $k = 0, 1, 2$, the imaginary part is positive. Thus

$$\int_0^{\infty} \frac{dx}{x^6+1} = 2\pi i \sum_{k=0}^{2} \operatorname*{Res}_{z=z_k} \frac{1}{z^6+1}.$$

The poles of $z^6 + 1$ are simple, we apply Lemma 33.1 to compute the residues:

$$\int_0^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}2\pi i \sum_{k=0}^{2} \frac{1}{6z^5}\Big|_{z=z_k} = \pi i \sum_{k=0}^{2} \frac{1}{6z_k^5}.$$

Since $z_k^6 = -1$ (these are roots of the equation $z^6 = -1$), we have that

$$\int_0^{\infty} \frac{dx}{x^6+1} = \pi i \sum_{k=0}^{2} -\frac{z_k}{6} = -\frac{\pi i}{6}(e^{\pi i/6} + e^{\pi i/2} + e^{5\pi i/6}) = -\frac{\pi i}{6} \cdot 2i = \frac{\pi}{3}.$$

34. April 1, April 3

Definition 34.1. *Let $f(z)$ be holomorphic in the region* $\mathrm{Im}(s) \geq 0$ *apart from finitely many isolated singularities. The* **Cauchy principal value** *of the integral*

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx$$

is

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)e^{ix}dx.$$

The Cauchy principal value equals the full integral precisely when the integral converges absolutely, that is,

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

34.1. **No real singularities.** Suppose that

$$\lim_{|z|\to\infty} f(z) = 0.$$

For $R > 0$ large, let $\gamma = \gamma_R \cup [-R, R]$, where γ_R is parametrized by $z(t) = Re^{it}$, $0 < t < \pi$. Then

$$\int_{-R}^{R} f(x)e^{ix}dx + \int_{\gamma_R} f(z)e^{iz}dz = \int_{\gamma} f(z)e^{iz}dz = 2\pi i \sum_{\substack{|z_k|\leq R\\ \mathrm{Im}(z_k)>0}} \operatorname*{Res}_{z=z_k} f(z).$$

Let

$$M(R) = \max_{0\leq\theta\leq\pi} |f(Re^{i\theta})|.$$

Since $\sin\theta$ is symmetric about $\pi/2$ on $[0, \pi]$, we have

$$\Big|\int_{\gamma_R} f(z)e^{iz}dz\Big| \leq M(R)\int_0^{\pi} e^{-R\sin\theta}Rd\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}Rd\theta$$

Note that

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1, \qquad 0 \leq \theta \leq \frac{\pi}{2},$$

as can be proved in a number of ways (geometry, Taylor series, etc.). Thus

$$2\int_0^{\pi/2} e^{-R\sin\theta}Rd\theta \leq 2R\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta = \pi.$$

Thus

$$\Big|\int_{\gamma_R} f(z)e^{iz}dz\Big| \leq M(R)\cdot\pi.$$

Since $\lim_{|z|\to\infty} f(z) = 0$, we have

$$0 \leq \lim_{R\to\infty}\Big|\int_{\gamma_R} f(z)e^{iz}dz\Big| \leq \pi \lim_{R\to\infty} M(R) = 0.$$

Thus

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)e^{ix}dx = 2\pi i \sum_{\mathrm{Im}(z_k)>0} \operatorname*{Res}_{z=z_k} f(z)$$

Example 34.2.

$$\int_0^\infty \frac{\cos x}{x^2+1}dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^\infty \frac{e^{ix}}{x^2+1}dx = \frac{1}{2}\mathrm{Re}\Big(2\pi i \operatorname*{Res}_{z=i}\frac{e^{iz}}{z^2+1}\Big) = \frac{1}{2}\mathrm{Re}\Big(\frac{2\pi i e^{iz}}{\frac{d}{dz}(z^2+1)}\Big|_{z=i}\Big) = \frac{\pi}{2e}.$$