Dynamical Ideals

Justin Young

Definition 1. *A dynamical ideal consists of a group* Γ*, its action on a set* X*, and an ideal* I *on* X *invariant under the group action. Dynamical ideals will be denoted by* $(\Gamma \curvearrowright X, I)$*.*

Definition 2. *Given a dynamical ideal* $(\Gamma \curvearrowright X, I)$*, one can construct a permutation model of set theory as follows: let* $V[[X]]$ *be a model of set theory with atoms using* X *as the atoms, and define the permutation model* $W[[X]]$ *to be the transitive part of* $\{A \in V[[X]] : \text{there exists } b \in I \text{ such that } pstab(b) \subseteq stab(A)\}$*.*

Certain dynamical properties of the dynamical ideal correspond to fragments of choice in the permutation model.

Definition 3. *A dynamical ideal* $(\Gamma \curvearrowright X, I)$ *is* σ*-complete if for all* $a \in I$ *and sequences* $(b_n : n \in \omega) \subseteq I$*, there exists group elements* $\gamma_n \in pstab(a)$ *such that* $\bigcup \gamma_n \cdot b_n \in I$*.*

Theorem 1. *If a dynamical ideal is* σ*-complete, then the associated permutation model satisfies the axiom of countable choice.*

Definition 4. *A dynamical ideal* $(\Gamma \curvearrowright X, I)$ *has cofinal orbits if for all* $a \in I$ *there exists* $b \in I$ *which is a-large: for all* $c \in I$ *there exists* $\gamma \in pstab(a)$ *such that* $c \subseteq \gamma \cdot b$*.*

Theorem 2. *If a dynamical ideal has cofinal orbits, then the corresponding permutation model satisfies the axiom of well-ordered choice.*

We look at some examples of dynamical ideals from topological spaces with these dynamical properties.

1 Ideal of countable compact sets

Definition 5. *Let* (X, τ) *be a topological space. We say a compact set* $A \subseteq X$ *is disjointable with respect to basis* $\mathcal{B}$ *if for any cover* $\mathcal{C}$ *of* A *there exists a refinement into pairwise disjoint elements of* $\mathcal{B}$*. Note that* A *is disjointable with respect to* τ *if and only if* A *has dimension 0.*

Definition 6. *Let* X *be a topological space with basis* $\mathcal{B}$*, and* I *an ideal on* X*. We call* I *disjointable if for any* $a \in I$ *and any cover* $\mathcal{C}$ *of* a *by open sets, there exists a refinement by pairwise disjoint basic open sets.*

Proposition 3. *Let* X *be Polish,* $A \subseteq X$ *be compact, let* $B \subseteq X$ *be countable, and let* $U \subseteq X$ *be a basic open ball containing* A*. Then there is a basic open ball* $A \subseteq V \subseteq \overline{V} \subseteq U$ *with the same center as* U*. In particular,* V *can be chosen such that* $bd\,(V) \cap B = \emptyset$*.*

Proof. Write $U = Ball(x, r)$ as a ball centered at x with radius r. Let $\delta = \inf\{\varepsilon : A \subseteq Ball(x, \varepsilon)\}$. Since A is closed, we note $A \not\subseteq Ball(x, \delta)$, and so $\delta \neq r$. Hence for any $\varepsilon \in (\delta, r)$, we can let $V = Ball(x, \varepsilon)$ and have $V \subseteq \bar{V} \subseteq U$. We observe that if two balls are constructed from the same center with different radii, then the boundaries are disjoint. Hence if the boundaries are equal, they must be empty. If such a ε exists such that the boundary is empty, then the boundary is trivially disjoint from b. Otherwise, we conclude that the continuum many possible values for ε allow for continuum many possible distinct pairwise disjoint boundaries, and by the pigeonhole principle at least one of these boundaries is disjoint from b. □

The following is similar to Proposition 1, but in particular is useful to argue that Polish spaces with a basis of rational balls is disjointable.

Proposition 4. *Let X be Polish, $A, B \subseteq X$ countable compact, and let $U \subseteq X$ be a metric open ball containing A. Then there is a metric open ball with rational radius V with the same center as U such that $A \subseteq V \subseteq \overline{V} \subseteq U$, and $bd(V) \cap B = \emptyset$.*

Proof. Let $U = Ball(x, r)$ and define a continuous function $f : B \to \mathbb{R}$ by $f(y) = d(x, y)$. Note that since B is countable compact, so is $f(B)$, and hence $f(B)$ is nowhere dense. Let $\delta = \inf\{\varepsilon : A \subseteq Ball(x, \varepsilon)\}$, and note that $\{s \in \mathbb{Q} : \delta < s < r \text{ and } s \notin f(B)\}$ is nonempty. Fix some s from this set, and note $V = Ball(x, s)$ works. □

Proposition 5. *Let X be Polish, $A \subseteq X$ be countable compact, and let $\varepsilon > 0$. There is a finite cover $\mathcal{C}$ of A such that the $C \in \mathcal{C}$ are pairwise disjoint basic open balls of radius at most ε. In particular, $\bigcup \mathcal{C} \subseteq Ball(A, \varepsilon)$.*

Proof. We proceed by induction on the Cantor-Bendixson rank of A. Let α be the least ordinal such that $A^{(\alpha)} \neq \emptyset$ and $A^{(\alpha+)} = \emptyset$. Then $A^{(\alpha)}$ contains no limit points; we write $A^{(\alpha)} = \{x_i : i \in \omega\}$, and we recursively construct balls $B_i = Ball(x_i, \varepsilon_i)$ with $\varepsilon_i \leq \varepsilon$ such that (a) for $j < i, B_i \cap B_j = \emptyset$, (b) for $j > i, x_j \notin B_i$, and (c) for all $x \in A$ and all $i \in \omega$, $x \notin \text{bd}(B_i)$. We note that property (c) can be obtained by Proposition 3 above, and (a) and (b) can be done since $A^{(\alpha)}$ contains no limit points. In the end, we let $A^* = A \setminus \bigcup B_i$, and we note A^* is closed of rank $\leq \alpha$. We let $\delta = \min\{d(a, b) : a \in A^*, b \in \bigcup B_i\}$. By compactness, this minimum exists, and it is not 0 since the B_i were chosen to not intersect A in its boundary. Hence by induction, we can cover A^* with disjoint open balls of radius smaller than δ, and these balls will also be disjoint from the B_i hence we get a cover of all of A as desired. □

Corollary 5.1. *Fix a Polish space X and a complete compatible metric d. This space with the basis of rational metric balls and the ideal of countable compact sets is disjointable.*

Proof. Let $A \subseteq X$ be countable compact and let $\mathcal{C}$ be a cover of A. Use Proposition 4 to replace $\mathcal{C}$ with a refinement such that for each $C \in \mathcal{C}$, $\text{bd}(C) \cap A = \emptyset$. Let $\varepsilon = \min\{d(a, b) : a \in A, b \in \text{bd}(C) \text{ for } C \in \mathcal{C}\}$. Now use Proposition 5 to construct a cover of A by balls of radius smaller than ε. We note that by the choice of ε, this cover is guaranteed to be a refinement of $\mathcal{C}$. □

Question 6. *Is there an example of a space that is disjointable for one basis yet not disjointable for another basis?*

Answer 6.1. *Euclidean space R^n with the ideal of compact 0-dimensional sets is disjointable if the basis is the collection of all path-connected open sets, yet not if the basis is the collection of all metric balls.*

Question 7. *What topologies and ideals are disjointable for all bases?*

Answer 7.1. *Trivially, the discrete topology is disjointable for any ideal and basis.*

Definition 7. *Let (Γ, X, I, d) be a dynamical ideal with a metric. We say the dynamical ideal is tight if for all $a, b \in I$, $\varepsilon > 0$, there is $\gamma \in pstab(a)$ such that $\gamma \cdot b \subseteq Ball(a, \varepsilon)$.*

Proposition 8. *Let X be a metric space, Γ the group of homeomorphisms on X, and I the ideal of countable compact sets. If (Γ, X, I) is tight, then it is σ-complete.*

Proof. Let $a \in I, b_n \in I$ for $n \in \omega$. Since (Γ, X, I) is tight, find $\gamma_n \in \text{pstab}(a)$ such that $\gamma_n(b_n) \subseteq Ball(a, 1/n)$. We claim that $A = a \cup \bigcup \gamma_n(b_n) \in I$. Clearly A is countable. Now let (x_k) be a sequence in A such that $x_k \to x \in X$. There are two cases. If $\forall n \exists N \forall M \geq N x_M \in Ball(a, 1/n)$. Then $x \in a \subseteq A$. On the other hand, if $\exists n \forall N \exists M \geq N x_M \notin Ball(a, 1/n)$, then pass to a subsequence x_{M_k} such that for all k, $x_{M_k} \notin Ball(a, 1/n)$. Then for all k, $x_{M_k} \in \bigcup_{i<n} \gamma_i(b_i)$ which is closed. □

Question 9. *There is a possible generalization of the above notion of tightness beyond metric spaces as follows: Γ, X, I is tight if for all $a \in I$ there exists a family of open sets O_n such that $\bigcap O_n = a$ and for all $a, b \in I$ and open set $O \supseteq a$ there exists $\gamma \in pstab(a)$ such that $\gamma \cdot b \subseteq O$. However, to get a generalization of the proposition, we'd need to find a proof that addresses closure, not just sequential closure.*

Proposition 10. *Let $\{C_1, \ldots, C_m\}$ be a collection of pairwise disjoint closed balls in $\mathbb{R}^n$. Then there is a set K such that $C_1, C_2 \subseteq K$ and for $3 \leq i \leq m$, $C_i \cap K = \emptyset$. Further, we can find K such that $K = h(B)$ is the image of the unit ball under some homeomorphism of $\mathbb{R}^n$.*

Proof. Let L be a straight line path from the center of C_1 to the center of C_2. Since the balls are all closed pairwise disjoint, we can find larger balls $D_3 \supseteq C_3, D_4 \supseteq C_4, \ldots, D_m \supseteq C_m$ such that $C_1, C_2, D_3, \ldots, D_m$ are all pairwise disjoint. If L passes through D_i, then replace L such that it traces the geodesic along the boundary of D_i. We note that by this construction, L does not have a knot, so there is a homeomorphism ψ of $\mathbb{R}^n$ that sends L to the first axis. That is, $\psi(L) \subseteq \{(x, 0, \ldots, 0) : x \in \mathbb{R}\}$. We construct a cylinder about L as follows: $cyl_\rho L := \psi^{-1}(\{(x, r_2, \ldots, r_n) : \sqrt{r_2^2 + \cdots + r_n^2} \leq \rho, (x, 0, \ldots, 0) \in \psi(L)\})$. In particular, we can choose ρ small enough such that $cyl_\rho L$ doesn't meet any of the $C_3, \ldots, C_m$. Then let $K = C_1 \cup cyl_\rho L \cup C_2$. □

Proposition 11. *Let A, B be open balls whose closure is contained in the interior of the annulus $S^n \times [0,1]$. There exists a homeomorphsim of the annulus h such that $h(A) \subseteq B$ and $h \upharpoonright S^n \times \{0,1\} = id$.*

Proof. Note: for the 2 dimensional annulus, we can turn this into a corresponding problem with a square by finding a path from the inner circle to the outer circle that avoids A and B. Unfortunately, this approach generalizes to the cylinder $S^1 \times [0,1]^n$, as opposed to the annulus $S^n \times [0,1]$. An alternate approach is as follows:

We let $A = Ball(x, r)$ be the ball of radius r centered at x and $B = Ball(y, s)$ be the ball of radius s centered at y. We construct the homeomorphism of the annulus in three steps: h_1 will be such that $h_1(A)$ has radius at most s, h_2 will be such that $h_2 h_1(x)$ has last coordinate equal to the last coordinate of y, and finally h_3 will be such that $h_3 h_2 h_1(x) = y$. If $r \leq s$ already, let h_1 be the identity. Otherwise, let $t > r$ be such that $cl(Ball(x, t))$ is contained in the interior of the annulus. Now let h_1 be a contraction of this ball fixing the boundary such that $h_1(A)$ is a ball of radius s. To get h_2, let $x = (x_1, \ldots, x_n, k)$ and $y = (y_1, \ldots, y_n, \ell)$. Note that there is a homeomorphism f of $[0,1]$ such that $f(k) = \ell$. Let h_2 be the homeomorphism of the annulus that fixes the first n coordinates and maps the last coordinate according to f. To get h_3, rotate the interior of the annulus accordingly. $\square$

Proposition 12. *The dynamical ideal $(Homeo(\mathbb{R}^n), \mathbb{R}^n, I$, where I is the ideal of countable compact sets, is tight.*

Proof. For $n = 1$, a different argument is necessary. For $n \geq 2$, the following works.

Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a countable cover of a by pairwise disjoint basic open balls of radius at most ε such that bd $(C_i) \cap b = \emptyset$. Note that we can replace b with $b \setminus \bigcup \mathcal{C}$ and replace a with $a \cup (b \cap \bigcup \mathcal{C})$. The new b is still compact, and by the choice of $\mathcal{C}$, the new a is also compact, and still covered by $\mathcal{C}$. Now choose a δ small enough such that $Ball(b, \delta)$ is disjoint from $\bigcup \mathcal{C}$, and find a cover $\mathcal{D} = \{D_1, \ldots, D_\ell\}$ of b by pairwise disjoint balls of radius at most δ. We use induction on ℓ to construct γ such that $\gamma(b) \subseteq \bigcup \mathcal{C}$ and $\gamma \upharpoonright a = \text{id}$. The base case $\ell = 0$ is trivial. Now for general ℓ, we start by finding γ_1 such that $\gamma_1(b \cap D_1) \subseteq \bigcup \mathcal{C}$ and γ_1 fixes all other points of a and b. Use Proposition 10 to find a set K containing C_1, D_1 that is the image of the unit ball by a homeomorphism h_1 of $\mathbb{R}^n$. Use Proposition 3 to find a ball $C' \subsetneq C_1$ containing $a \cap C_1$, and note that $\bar{C}'$ is also the image of the unit ball by a homeomorphism h_2 of $\mathbb{R}^n$. Note that $h = h_2 \circ h_1$ is a homeomorphism of $\mathbb{R}^n$ that maps K into $\bar{C}'$, which is contained in the interior of K. Hence by the annulus theorem, the difference $K \setminus h(\text{int}(K))$ is homeomorphic to an annulus. From here, we use Proposition 3 to find a $D' \subsetneq D_1$ that contains $b \cap D_1$ and find some open ball $U \subset C_1 \setminus C'$. By Proposition 11, there is a homeomorphism of the annulus that takes D' to U while fixing the boundary. Hence we can define γ_1 to be this map inside the annulus and the identity outside the annulus to get a homeomorphism of $\mathbb{R}^n$ that fixes a and moves $b \cap D_1$ within ε of a. Now replace a with $a \cup \gamma_1(b \cap D_1)$, and replace b with $b \setminus D_1$. Note that a is still compact and covered by $\mathcal{C}$. Use the induction hypothesis to get γ_2 that maps b into $\bigcup \mathcal{C}$. Taking the composition, $\gamma_2 \circ \gamma_1$ is the desired homeomorphism. □

Corollary 12.1. *The dynamical ideal* $(Homeo(\mathbb{R}^n), \mathbb{R}^n, I$*, where* I *is the ideal of countable compact sets, is* σ*-complete.*

Proposition 13. *The dynamical ideal* $(Homeo(2^\omega), 2^\omega, I)$*, where* I *is the ideal of countable compact sets, is tight.*

Proof. Let $a, b \subseteq 2^\omega$ and ε be given. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a cover of a by balls of radius smaller than ε. Now choose a δ small enough such that $Ball(b, \delta)$ is disjoint from $\bigcup \mathcal{C}$, and let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be a cover of b by disjoint balls of radius smaller than δ. We argue by induction on m: find a basic open set $U \subseteq C_1$ such that $\bar{U} \cap a = \emptyset$, and let γ_1 be a homeomorphism that swaps D_1 and U while fixing all other points. By induction, find a γ_2 that maps $b \setminus D_1$ into $\bigcup \mathcal{C}$ and fixes $a \cup \gamma_1(b \cap D_1)$. Then we see $\gamma_2 \circ \gamma_1$ is a homeomorphism of 2^ω that fixes a and maps b within ε of a. □

Corollary 13.1. *The dynamical ideal* $(Homeo(2^\omega), 2^\omega, I)$ *is* σ*-complete.*

Question 14. *If* X *is a metric space and* I *is a* σ*-ideal of closed sets, then must the dynamical ideal* (Γ, X, I) *be tight?*

2 Ideal of closed nowhere dense sets

Definition 8. *Let* (Γ, X, I) *be a dynamical ideal. The ideal has cofinal orbits if for every* $a \in I$ *there is* $b \in I$ *which is a-large: for every* $c \in I$ *there is* $\gamma \in pstab(a)$ *such that* $c \subseteq \gamma \cdot b$*.*

Proposition 15. *Let* $X = \mathbb{R}^n$*,* Γ *be the group of homeomorphisms of* $\mathbb{R}^n$*, and* I *the ideal generated by closed nowhere dense sets. Then* (Γ, X, I) *has cofinal orbits.*

We first define the Sierpiński carpet and note a couple theorems about it because the carpet will be instrumental in finding an a-large set.

Definition 9. *Let* X *be a compact connected metric space. We say* X *is an* n*-dimensional Sierpiński carpet (S-carpet) if it can be embedded in the sphere* S^{n+1} *in such a manner that*

1. *the set* $\{U_i : i \in \omega\}$ *of components of* $S^{n+1} \setminus X$ *forms a sequence such that* $diam(U_i) \to 0$
2. $S^{n+1} \setminus U_i$ *is an* $n+1$ *cell for each* i*.*
3. *the set of closures* $\{\overline{U_i} : i \in \omega\}$ *is pairwise disjoint.*
4. $\overline{\bigcup U_i} = S^{n+1}$*.*

Any homeomorphic image of an S-carpet is also an S-carpet.

Definition 10. *We note that if $X \subseteq \mathbb{R}^{n+1}$ is an n dimensional S-carpet, then exactly one of the components of $\mathbb{R}^n \setminus X$ is unbounded. Given an $n+1$ cell $D \subseteq \mathbb{R}^{n+1}$, we say X fills D if the unbounded component of $\mathbb{R}^{n+1} \setminus X$ is exactly $\mathbb{R}^{n+1} \setminus D$. Equivalently, X fills D if $X \subseteq D$ and $bd(D) \subseteq X$.*

The following theorem is attributed to Whyburn for $n = 1$ and Cannon for $n \geq 2$. Cannon used the annulus theorem to extend Whyburn's results for all $n \neq 4$ since the annulus theorem had not been proved for $n = 4$ at the time. The annulus theorem has since been proven for the $n = 4$ case, and so we note the characterization theorem holds in all dimensions.

Proposition 16. *Let X and Y be two n-dimensional S-carpets embedded in S^{n+1} and let U and V be components of $S^{n+1} \setminus X$ and $S^{n+1} \setminus Y$ respectively. If h is a homeomorphism from the boundary of U to the boundary of V, then h can be extended to a homeomorphism from X to Y.*

This proposition tells us that all S-Carpets of the same dimension are homeomorphic. We note the following special case of the previous proposition:

Corollary 16.1. *Let X and Y be two $n-1$-dimensional S-carpets that fill the n cell $D \subseteq \mathbb{R}^n$. There is a homeomorphism φ of $\mathbb{R}^n$ which is the identity on $\overline{\mathbb{R} \setminus D}$ and whose restriction to X is a homeomorphism from X to Y.*

Proof. Apply the previous proposition such that h is the identity map on the boundary of D. The proposition yields an extension $\bar{h} : X \to Y$. It remains to extend $\bar{h}$ to φ by defining φ on each component U_i of $\mathbb{R} \setminus X$. The unbounded component is exactly $\mathbb{R} \setminus D$, and we let φ be the identity here. Otherwise, for the bounded U_i, note that $\bar{h}(bd(U_i))$ is the boundary for one of the components of $D \setminus Y$. Denote the corresponding component of $D \setminus Y$ by V_i, and let h_i extend $\bar{h} \restriction bd(U_i)$ be a homeomorphism between $\overline{U_i}$ and $\overline{V_i}$. Take φ to be the union of $\bar{h}$ and all h_i. □

When he first introduced the one-dimensional S-carpet, Sierpiński proves that his construction contains a topological image of a any compact nowhere dense set in the plane. His theorem generalizes to higher dimensions, but the homeomorphism given is a product of homeomorphisms and hence is not the identity on the boundary of the square that his carpet fills. We state a slightly modified version of his theorem, generalized to arbitrary finite dimension.

Proposition 17. *Let $X = [0, 1]^n$, let $A \subseteq X$ be compact nowhere dense. There is an S-Carpet K which fills X such that $A \subseteq K$.*

Proof. We first define some notation: for the entirety of this proof, a box will be taken to mean a product of closed intervals. We consider arbitrary elements of X to have the form $x = (x_0, x_1, \dots, x_{n-1})$ and we denote sequences in $(3^n)^{<\omega}$ as $(\alpha_i^j : j \in k, i \in n) = (\alpha_0^0\alpha_1^0 \dots \alpha_{n-1}^0; \alpha_0^1 \dots \alpha_{n-1}^1; \dots; \alpha_0^{k-1} \dots \alpha_{n-1}^{k-1})$. We will define $D(\alpha_i^j : j \in k, i \in n)$ (referred to as $D_k(\alpha_i^j)$ or sometimes just by $D(\alpha_i^j)$) by recursion on k, and in the end define

$$K = X \setminus \bigcup_{(\alpha_i^j)\in(3^n)^{<\omega}} \mathrm{int}(D(\alpha_i^j)).$$

To start with the case $k = 0$, divide X into 3^n congruent boxes separated by the hyperplanes $x_i = 1/3$ and $x_i = 2/3$. Denote the box containing the center by V_0, and use the fact that A is nowhere dense to find a box $D_0 \subseteq V_0$ such that $A \cap D_0 = \emptyset$. D_0 is a product of intervals; say the interval in the i'th coordinate is $[x_i^1, x_i^2]$. Extend the hyperplanes $x_i = x_i^1$ and $x_i = x_i^2$ to get new boxes, and label the boxes by $R_1(\alpha_i^j)$ such that $R_1(\alpha_i^j)$ denotes the α_i^0'th box along the i'th axis. Formally, we define $R_1(\alpha_i^j) = \prod_{i\in n}[x_i^{\alpha_i^0}, x_i^{\alpha_i^0+1}]$, where $x_i^0 = 0$ and $x_i^3 = 1$. We note that $R_1(1, \dots, 1) = D$ and that for all possible intervals, $[x_i^{\alpha_i^0}, x_i^{\alpha_i^0+1}] < 2/3$. In particular, we see $diam(R_1(\alpha_i^j)) < \sqrt{n} \cdot 2/3$.
Now for successor values of k, assume that for $(\alpha_i^j) \in (3^n)^k$, we have already defined all of the boxes $R_k(\alpha_i^j)$ and that $diam(R_k(\alpha_i^j)) < \sqrt{n} \cdot (2/3)^k$. Fix a particular sequence (α_i^j), and we define $D_k(\alpha_i^j)$ by first dividing $R_k(\alpha_i^j)$ into 3^k congruent boxes, denoting the box which contains the center of $R_k(\alpha_i^j)$ by $V_k(\alpha_i^j)$. Let $D_k(\alpha_i^j) \subseteq V_k(\alpha_i^j)$ be a box disjoint from A. Now for $\beta_i^j : j \in k+1, i \in n$ which extend α_i^j, define $R_{k+1}(\beta_i^j)$ to be the boxes obtained by extending the hyperplane edges of $D_k(\alpha_i^j)$. Again we have $R_{k+1}(\alpha_i^j; 1, \dots, 1) = D_k(\alpha_i^j)$, and in every dimension, $R_{k+1}(\beta_i^j)$ is at most 2/3 the length of $R_k(\alpha_i^j$, so we see $diam(R_{k+1}(\beta_i^j)) < \sqrt{n} \cdot (2/3)^{k+1}$.
We let K have the definition stated above, and now we show that K is an S-Carpet which contains A. By construction, each $D(\alpha_i^j)$ is disjoint from A, so K contains A. We note that not all of the $\mathrm{int}(D(\alpha_i^j))$ are full components of $X \setminus K$; in particular if $(\alpha_i^j) = (\beta_i^j)^\frown(1, \dots, 1)^\frown(\gamma_i^j)$ contains the sequence of n 1's, then $D(\alpha_i^j)$ is a proper subset of $D(\beta_i^j)$. Otherwise, we consider the sequence of complementary components $\{U_i : i \in \omega\}$ to be the interiors of maximal $D(\alpha_i^j)$. Since $diam(D_k(\alpha_i^j)) = diam(R_{k+1}(\alpha_i^j; 1, \dots, 1) < \sqrt{n} \cdot (2/3)^{k+1}$, we see that $diam(U_i) \to 0$. Further, by embedding K into the sphere by identifying the boundary of X, we see that for each i, $S^{n+1} \setminus U_i$ is indeed an $n+1$ cell. By construction, the maximal $D(\alpha_i^j)$ are disjoint, hence so are the closures of the U_i. Finally, to see that K itself is nowhere dense, let O be an arbitrary open set. Note that for a fixed k, the set $\{R_k(\alpha_i^j) : \alpha_i^j \in (3^n)^k\}$ covers X. Choose a large enough k and $(\alpha_i^j) \in (3^n)^k$ such that $R_k(\alpha_i^j) \subseteq O$, and see $D_k(\alpha_i^j) \subseteq R_k(\alpha_i^j) \subseteq O$. □

Now we are ready to prove that the ideal of closed nowhere dense sets has cofinal orbits.

Proof. Let $A \subseteq \mathbb{R}^n$ be closed nowhere dense. First let L be the union of all hyperplanes of the form $x_i = m$ for $m \in \mathbb{Z}$, and let $K \supseteq L \cup A$ be closed nowhere dense such that the components of its complement are all homeomorphic to open balls. In particular, note that we can do this by filling each cube Q of L by a S-Carpet which contains $A \cap Q$.

Now index the components of $\mathbb{R}^n \setminus K$ by $\{U_i : i \in \omega\}$ and fill each $\overline{U_i}$ with an S-Carpet B_i. We let $B = K \cup \bigcup B_i$, and claim that $B \in I$ is A-large.

To see that B is closed, we note that each component of the complement of B is an open set, and to see that B is nowhere dense, let U be an arbitrary open set. There is some U_i that meets U, and we note that some component of $\mathbb{R}^n \setminus B_i$ is contained in $U \cap U_i$. Finally, to see that B is A-large, let C be a closed nowhere dense set. Let $C_i = C \cap U_i$ and let D_i be a S-Carpet filling $\overline{U_i}$ which contains C_i. Since both B_i and D_i are S-Carpets filling $\overline{U_i}$, we can use Corollary 16.1 to find a homeomorphism $h_i : \overline{U_i} \to \overline{U_i}$ which is the identity on the boundary and such that $h_i(B_i) = D_i$. In particular, we also get $h_i(B_i) \supseteq C_i$. After defining h_i in this manner for each i, we let $\gamma = id \restriction K \cup \bigcup_{i\in\omega} h_i$ be the desired homeomorphism to witness that B is A-large. We note that since each h_n is the identity on the boundary of where it's defined, and this boundary is exactly the portion of K on which h_n is defined, γ is well-defined. Since γ is the identity on $K \supseteq A$, we know $\gamma \in \mathrm{pstab}(A)$, and by construction we have $C \subseteq \gamma(B)$. □

As a note, we get the same result working in S^n and/or by using the ideal of compact nowhere dense sets instead of closed nowhere dense.

3 Ideal of compact 0-dimensional sets

In this section, we let $X = \mathbb{R}^2$, Γ be the group of self-homeomorphisms of $\mathbb{R}^2$ acting by application, and I be the ideal generated by compact 0-dimensional sets.

Theorem 18. *The dynamical ideal $(\Gamma \curvearrowright X, I)$ has cofinal orbits.*

Proof. We note that if $A \subseteq \mathbb{R}^2$ is compact nowhere dense, then there exists a map $f : [0,1] \to \mathbb{R}^2$ which is homeomorphic on its image such that $A \subseteq f''[0,1]$. Hence without loss of generality, we can assume that a is contained within the unit circle, and since closed sets of dimension 0 are nowhere dense in $\mathbb{R}$ and all nowhere dense sets are contained in an $n-1$ dimensional S-carpet, we can assume that a is the 0-dimensional S-carpet, i.e. the Cantor set. We fill each component U_i of $S^1 \setminus a$ with another copy of the Cantor set denoted b_i, and claim $b = a \cup \bigcup b_i$ is a-large.

Let $c \in I$ be given; we construct the necessary homeomorphism in three steps. Now find a $\gamma_1 \in \Gamma$ that fixes S^1 pointwise such that $c \setminus a$ is disjoint from the rays which extend from the origin and pass through points of a. Let V_i be the open set consisting of points which lie on the same ray from the origin as some point in U_i, and let $c_i = \gamma_1 \cdot c \cap V_i$. Now let p, q denote the endpoints of U_i, and find a path from p to q which stays in V_i and contains every element of c_i. Let φ_i be the selfhomeomorphism of V_i which fixes the boundary of V_i and maps this path to U_i, and let γ_2 be the union of all φ_i. Finally, we note that on each U_i there is a boundary-preserving self-homeomorphism ψ_i such that $b_i \supseteq \psi_i \varphi_i \cdot c_i$. We note that ψ_i can be extended to a boundary-preserving self-homeomorphism $\bar{\psi}_i$ of V_i, and we let γ_3 be the union of all $\bar{\psi}_i$. It follows that $\gamma_3\gamma_2\gamma_1$ is the desired homeomorphism to witness that b is a-large. □

We note that this result also works when I is the ideal of closed 0-dimensional sets - a slightly modified argument shows the compactness assumption is not necessary. The result may also hold when X is a higher-dimensional Euclidean space, although one needs to be careful of knots which might impede the argument.

4 Urysohn Rational Ultrametric Space

Definition 11. *Consider the class K of finite rational-valued ultrametric spaces. This forms a Fraïssé class, and we let X denote the Fraïssé limit of K.*

Definition 12. *For $r \in \mathbb{R}$, we define $I_r \subseteq \mathcal{P}(X)$ such that $A \in I_r$ if and only if for all $x \in X,\ 0 < \delta < \varepsilon \leq r$, there exists $y \in Ball(x, \varepsilon)$ such that $Ball(y, \delta) \cap A = \emptyset$.*

This definition is a bit awkward since it quantifies over all $\varepsilon \leq r$. However, using ε helps in the proofs of Proposition 20 and Proposition 21.

As a note, if we want to check that $A \in I_r$, then it is enough to quantify x only over A itself. If $Ball(x, \varepsilon)$ is disjoint from A, then finding a point y is trivial, and if $a \in Ball(x, \varepsilon) \cap A$, then $Ball(a, \varepsilon) = Ball(x, \varepsilon)$ by properties of the ultrametric.

Proposition 19. *Let $\delta < r$. Then I_r does not contain any δ-nets of X.*

Proof. Let $\delta < r$ and A be a δ-net. To see that $A \notin I_r$, note that for every choice of $y \in X$, $Ball(y, \delta)$ has nontrivial intersection with A. □

Proposition 20. *If $s \leq r$, then $I_r \subseteq I_s$.*

Proof. Let $A \in I_r$. Choose $x \in X$ and δ, ε such that $0 < \delta < \varepsilon \leq s$. Since $s \leq r$, we see $\varepsilon \leq r$, and hence we can find a y to witness $A \in I_r$. We note this same y witnesses that $A \in I_s$. □

Proposition 21. *I_r is an ideal.*

Proof. Note that if $A \in I_r$ and $B \subseteq A$, then any choice of y to witness $A \in I_r$ also witnesses $B \in I_r$ since $Ball(y, \delta) \cap B \subseteq Ball(y, \delta) \cap A = \emptyset$. To see that I_r is closed under unions, let $A, B \in I_r$. Given $x \in X$ and $0 < \delta < \varepsilon \leq r$, let δ' be such that $\delta < \delta' < \varepsilon$. Find $y_1 \in Ball(x, \varepsilon)$ such that $Ball(y_1, \delta')$ is disjoint from A. Next find $y_2 \in Ball(y_1, \delta')$ such that $Ball(y_2, \delta)$ is disjoint from B. Note that $y_2 \in Ball(x, \varepsilon)$ and that $Ball(y_2, \delta) \subseteq Ball(y_1, \delta')$ is also disjoint from A. □

Proposition 22. *I_r contains all singletons and is invariant under isometries.*

Proof. Let $A = \{x\} \subseteq X$ and $0 < \delta < \varepsilon \leq r$. Let δ' be rational such that $\delta < \delta' < \varepsilon$, and find y such that $d(x, y) = \delta'$. To see that I_r is invariant under isometries, let $A \in I_r$ and given x, δ, ε, let y as needed such that $Ball(y, \delta) \cap A = \emptyset$. For any isometry φ and given $\varphi(x), \delta, \varepsilon$, we see $\varphi(y)$ will work such that $Ball(\varphi(y), \delta) \cap \varphi(A) = \emptyset$. □

Proposition 23. *I_r contains infinite sets. In particular, I_r contains all closed sequences.*

Proof. Let $x_n \to x \in X$, and let $A = \{x, x_0, x_1, \dots\}$. Given $0 < \delta < \varepsilon \leq r$ and some $a \in A$, we must find a suitable y. We first note that if $x \notin Ball(a, \varepsilon)$, then $Ball(a, \varepsilon)$ contains only finitely many points of A, and so such a choice of y can be found. Otherwise, it is sufficient to consider $Ball(x, \varepsilon)$ and we note that there are finitely many x_i such that $\delta < d(x_i, x) < \varepsilon$. We let δ' be rational such that for all of those x_i, we have $d(x_i, x) < \delta' < \varepsilon$. Now find some y such that $d(y, x) = \delta'$. By properties of the ultrametric, we see that for all $x_i \in Ball(x, \varepsilon)$, we have $d(y, x_i) = \delta' > \delta$. Hence $Ball(y, \delta) \cap A = \emptyset$. □

The ideal of closed sets with finitely many accumulation points is not σ-complete in any topological space which contains infinitely many accumulation points. To see this, let b_i be sets with i-many accumulation points. Then regardless of what homeomorphisms φ_i are chosen, $\bigcup \varphi_i b_i$ has infinitely many limit points.