How to use Topology to Study Set Theory

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Set Theory

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If there exists a model M that satisfies A and B as well as a model N that satisfies $\neg A$ and B, then we can say that A is independent of B.

Choice

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The **Axiom of Countable Choice** is the statement that every countable family of nonempty sets has a choice function.

Set Theory with Atoms

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ZFCA denotes ZFA+Axiom of Choice

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In the end, $V[[X]] = \bigcup_{\alpha\in Ord} V_\alpha[[X]]$

$V \subset V[[X]]$ is the class of sets whose transitive closure contains no atoms. V is called the pure part of $V[[X]]$ and is itself a model of ZF.

Group Actions

Definition

Let $\Gamma \curvearrowright X$ be a group action. For $x \in X$, define $stab(x) = \{\gamma \in \Gamma : \gamma \cdot x = x\}$. For $a \subseteq X$, define $pstab(a) = \{\gamma \in \Gamma : \forall x \in a \ \gamma \cdot x = x\}$.

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Note that V, the pure part of $V[[X]]$ is fixed pointwise by the action.

Dynamical Ideal

Definition

A dynamical ideal is a tuple $(\Gamma \curvearrowright X, I)$ where Γ is a group, X is a set which Γ acts on, and I is an ideal on X which contains all singletons and is invariant under the group action (i.e. $\Gamma \cdot I \subseteq I$).

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Definition

An ideal on X is a set $I \subseteq \mathcal{P}(X)$ such that

1. $A, B \in I \implies A \cup B \in I$
2. $A \in I, B \subseteq A \implies B \in I$

Examples of Dynamical Ideals

Let (X, τ) be a topological space, let Γ be the group of homeomorphisms of X, acting by application (i.e. $\gamma \cdot x = \gamma(x)$). Then any ideal defined solely in terms of the topology will yield a dynamical ideal:

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Definition

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2. $a \subseteq X$ is totally disconnected if the only connected components of a are singletons.

Permutation Model from Dynamical Ideal

Definition

The permutation model associated with the dynamical ideal $(\Gamma \curvearrowright X, I)$ is the transitive part of $\{A \in V[[X]] | \exists b \in I \, pstab(b) \subseteq stab(A)\}$. The permutation model is denoted $W[[X]]$.

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Definition

By "transitive part" we mean that we want $A \in W[[X]] \implies A \subset W[[X]]$.

The Permutation Model

Theorem

Given a dynamical ideal $(\Gamma \curvearrowright X, I)$*, the associated permutation model* $W[[X]]$ *is a model of ZFA. Except in trivial cases, it will not satisfy the full Axiom of Choice.*

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Lemma

We have $I, X \in W[[X]]$, *and* $V \subset W[[X]]$.

σ-Complete Ideals

Definition

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Definition

Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. The dynamical ideal is dynamically σ-complete if for every set $a \in I$ and every countable sequence $(b_n : n \in \omega) \subseteq I$ there are group elements $\gamma_n \in \mathrm{pstab}(a)$ such that $\bigcup_n \gamma_n \cdot b_n \in I$.

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Theorem

(Zapletal, 2024) Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. If the dynamical ideal is dynamically σ-complete, then the associated permutation model satisfies the axiom of countable choice.

A Tool for σ-completeness

Definition

Let $(Homeo(X) \curvearrowright X, I)$ be a dynamical ideal. We say the dynamical ideal is tight if for all $a, b \in I$ and for all open $U \subseteq X$, there is $\gamma \in pstab(a)$ such that $\gamma \cdot b \subseteq U$.

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Proposition

(Y.) Let I be the ideal generated by countable closed sets on a topological space X that is G_δ, normal Hausdorff, and sequential. If the dynamical ideal $(Homeo(X) \curvearrowright X, I)$ is tight, then it is dynamically σ-complete.

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Proof.

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A similar argument shows the result holds when $X = \omega^\omega$.

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4. Given $C \in \mathcal{C}$ and $D \in \mathcal{D}$, find a set K which is the image of the unit circle under a self-homeomorphism of $\mathbb{R}^n$, contains C and D and does not meet any other set in either cover.

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5. Apply the Annulus Theorem to the region obtained in the previous step.

Modifications

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Given $a, b \in I$, tile X with homeomorphic copies of $[0,1]^n$ such that the boundaries avoid a, b. Now deal with each cube individually. $\square$

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Proof.

Use a sphere to separate the cube into two parts which can be dealt with individually. □

A Non-example

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1. Let $a = \emptyset$ and for each n let b_n be a $1/n$ net of X.

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Proof.

1. Let $a = \emptyset$ and for each n let b_n be a $1/n$ net of X.
2. Note that regardless of the choice of γ_n, $\bigcup \gamma_n \cdot b_n$ will be a dense set.

□

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Definition

Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal. It has cofinal orbits if for every $a \in I$ there exists $b \in I$ which is a-large: for every $c \in I$ there exists $\gamma \in pstab(a)$ such that $c \subseteq \gamma \cdot b$.

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Theorem

(Zapletal, 2024) Let $(\Gamma \curvearrowright X, I)$ be a dynamical ideal with cofinal orbits. The corresponding permutation model satisfies the axiom of well-ordered choice.

An example of cofinal orbits

Example

(Y., independently discovered by M. Elekes) Let $X = [0,1]^n$, and I the ideal generated by closed nowhere dense sets. Then $(Homeo(X) \curvearrowright X, I)$ has cofinal orbits.

An example of cofinal orbits

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The argument involves building a Sierpiński carpet on top of nowhere dense sets:

Sierpiński Carpet

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$A \subseteq X = [0,1]^n$ is an $n-1$-dimensional Sierpiński carpet filling X if

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$A \subseteq X = [0,1]^n$ is an $n-1$-dimensional Sierpiński carpet filling X if

1. A is closed nowhere dense
2. $bd(X) \subseteq A$
3. The set of components of $X \setminus A$, $\{U_i : i \in \omega\}$ is such that $diam(U_i) \to 0$ and each U_i is homeomorphic to an open ball.
4. $\{\overline{U_i} : i \in \omega\}$ is pairwise disjoint

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The following is a corollary of theorems of Whyburn ($n = 2$, 1958) and Cannon ($n \geq 3$, 1972):

Lemma

Given Sierpiński carpets $A, B \subseteq X$, there is a self-homeomorphism of X such that $h(A) = B$ and $h \restriction bd(X) = id$.

Sierpiński Carpet

Lemma

Given $A \subseteq [0,1]^n$ nowhere dense, there is a Sierpiński Carpet B filling $[0,1]^n$ such that $A \subseteq B$.

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Given $A \subseteq [0,1]^n$ nowhere dense, there is a Sierpiński Carpet B filling $[0,1]^n$ such that $A \subseteq B$.

Proof.

Fix a countable dense subset $\{x_i : i \in \omega\}$ and construct U_i centered at x_i such that

1. $diam(U_i) < 1/i$
2. $\overline{U_i}$ is disjoint from A and $\overline{U_j}$ for $j < i$.

Sierpiński Carpet

Lemma

Given $A \subseteq [0,1]^n$ nowhere dense, there is a Sierpiński Carpet B filling $[0,1]^n$ such that $A \subseteq B$.

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1. $diam(U_i) < 1/i$
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In the end, let $B = [0,1]^n \setminus \bigcup U_i$. □

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Proof.

1. Given $a \in I$, let $K \supseteq a$ be a Sierpiński carpet filling $[0,1]^n$.

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4. Finally, paste all of the γ_i together.

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Let X be any manifold and $\Gamma = Homeo(X) \curvearrowright X$ by application. Then for $I = \{a \subseteq X : \overline{a} \text{ is nowhere dense}\}$, the dynamical ideal has cofinal orbits.

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(Moise) Let M and M' be totally disconnected compact sets in $\mathbb{R}^2$, and let $\varphi : M \to M'$ be a homeomorphism. φ extends to a homeomorphism $\overline{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$.

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Corollary

(Moore-Kline theorem) Every totally disconnected compact set in $\mathbb{R}^2$ lies in an arc in $\mathbb{R}^2$.

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4. In the end, let $\varphi = \bigcup_{n \in \omega} \varphi_n$.

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4. Let $\psi_i : U_i \to U_i$ be a boundary preserving homeomorphism such that $\psi_i(\overline{\varphi}(c) \cap U_i) \subseteq b_i$. Let $\psi = \bigcup_{i \in \omega} \psi_i$.

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5. Extend ψ to $\overline{\psi} : \mathbb{R}^2 \to \mathbb{R}^2$. Note $\overline{\psi} \circ \overline{\varphi}$ works.

$\square$

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5. What other structures should we use to study set theory?

Thank you!