LECTURE 2

AGENDA:

1. Basic Set Theory
2. Sample spaces and events
3. Probability

BASIC SET THEORY

Def: A set is a collection of ~~[illegible]~~ distinct objects, which are called elements or points of the set.

- Sets are denoted by capital letters $A, B, C, \ldots$
- Notation: "$A \subset B$": A is a subset of B, i.e., all points in A are also in B

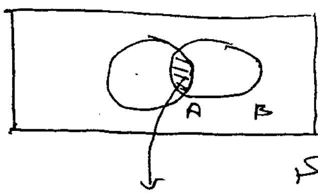
  "ϕ" denotes the null or empty set, contains no points
- Union "$A \cup B$": Collection of points which are in A or B
- Intersection: "$A \cap B$": Collection of points which are in A and B.

- S denotes the universal set, i.e., collection of all points of interest in the current situation

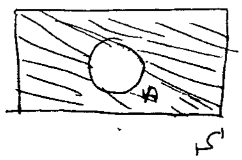
- <u>Complement:</u> "$\overline{A}$" : Collection of all points in S that is not in A

- A and B are called <u>mutually exclusive</u> if $A \cap B = \phi$ (empty set)

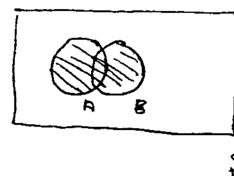
Venn Diagrams are graphical ways of representing sets.



Shaded region: $A \cap B$



Shaded region : $\overline{A}$



Shaded region: $A \cup B$

<u>Distributive laws:</u>

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

<u>De Morgan's Laws:</u>

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

These laws or rules will be useful in probability computations.

Example: $S = \{1, 2, 3, 4\}$

$A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{4\}$

$A \cup B = \{1, 2, 3, 4\}$
$A \cap B = \{2, 3\}$
$A \cap C = \phi$
$C \subset B \Rightarrow C \cup B = B, \; C \cap B = C$
$\overline{A} = \{4\}$

Homework: Verify the distributive laws and De Morgan's laws for A, B, C above.

SAMPLE SPACES AND EVENTS

- Perform a random experiment / observe a random phenomenon. For example, consider the tossing of a coin 4 times or percentage of a population affected by an epidemic.

- Def: The SAMPLE SPACE S of a random experiment is the set of all possible outcomes of the experiment listed in a mutually exclusive and exhaustive way

Examples: • Toss a coin 4 times

$S = \{HHHH, HHHT, HHTT, ---------\}$

In total $2^4 = 16$ possible outcomes.

This is an example of a discrete or countable sample space.

• Percentage of population affected by an epidemic

$S = [0, 100] \rightarrow$ All real numbers from 0 to 100.

This is an example of a continuous or uncountable sample space.

Def: An EVENT is any collection of sample points. In other words, any subset of the sample space S is called an event.

Example: Toss a coin 3 times

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

A = Event that there is atleast one heads

$\Rightarrow A = \{HHH, HHT, HTH, THH, TTH, THT, HTT\}$

B = Event that there is atmost one heads

$B = \{HTT, THT, TTH, TTT\}$

FORMAL DEFINITION OF PROBABILITY

Intuitively, "probability" of an event is number between 0 and 1 expressing our belief in the occurrence of the event in a single performance of an experiment.

$\mathcal{S}$ = Sample space of a random experiment
$\mathcal{A}$ = Collection of all possible events

__Def:__ A PROBABILITY ASSIGNMENT P for a random experiment is a numerically valued function that assigns a value $P(A)$ to every event A so that the following axioms are satisfied:

1) $P(A) \geq 0$ for every event A
2) $P(\mathcal{S}) = 1$
3) If $A_1, A_2, A_3, \ldots\ldots$ is a sequence of mutually exclusive events (i.e., $A_i \cap A_j = \phi$ for every $i \neq j$), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

- $\boxed{P(\phi) = 0}$. Choose $A_1 = \mathcal{S}$ and all others to be ϕ in 3).
- If $\boxed{A \cap B = \phi}$, then $\boxed{P(A \cup B) = P(A) + P(B)}$. Choose $A_1 = A$, $A_2 = B$ and all others to be ϕ in 3).
- If $\boxed{A \subset B}$, then $\boxed{P(A) \leq P(B)}$.

DEFINING AND CALCULATING THE PROBABILITY OF AN EVENT BY THE SAMPLE POINT METHOD (DISCRETE SAMPLE SPACE)

1) DEFINE THE EXPERIMENT.
2) CONSTRUCT THE SAMPLE SPACE.
3) ASSIGN PROBABILITIES TO EACH OF THE SAMPLE POINTS, MAKING SURE THEY ADD UP TO 1.
4) EXPRESS EVENT OF INTEREST AS A COLLECTION OF SAMPLE POINTS.
5) FIND P(A) BY SUMMING THE PROBABILITIES OF SAMPLE POINTS IN A.

FOUR RULES/LAWS WHICH FOLLOW FROM THE DEFINITION OF PROBABILITY:

- $P(\phi) = 0$
- $P(A \cup B) = P(A) + P(B)$ IF A & B ARE DISJOINT, i.e. $A \cap B = \phi$
- $P(\bar{A}) = 1 - P(A)$
- IF $A \subset B$, THE $P(A) \leq P(B)$

$\boxed{\text{Proof of } P(\phi)=0}$ For axiom 3), choose

$A_1 = S$, $A_i = \phi$ for all $i \geq 2$. It is easy to verify that these events are mutually exclusive. Hence, we get that,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\Rightarrow P(S) = P(S) + \sum_{i=2}^{\infty} P(A_i), \text{ since } \bigcup_{i=1}^{\infty} A_i = S.$$

$$\Rightarrow \sum_{i=2}^{\infty} P(\phi) = 0.$$

Since $P(\phi) \geq 0$, by axiom 1), it follows that $P(\phi) = 0$.

$\boxed{\text{Proof of } P(A \cup B) = P(A) + P(B) \text{ if } A \cap B = \phi}$

For axiom 3), choose $A_1 = A$, $A_2 = B$, $A_i = \phi$ for all $i \geq 3$. It is easy to verify that these events are mutually exclusive. Hence, we get that,

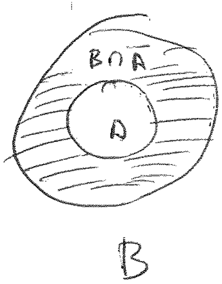
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) + \sum_{i=3}^{\infty} P(\phi)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B), \text{ since } P(\phi) = 0.$$

$\boxed{\text{Proof of } P(\bar{A}) = 1 - P(A)}$ Since $A \cap \bar{A} = \phi$, $P(A \cup \bar{A}) = P(A) + P(\bar{A})$. But $A \cup \bar{A} = S$, Hence $P(A) + P(\bar{A}) = P(A \cup \bar{A}) = P(S) = 1 \Rightarrow P(\bar{A}) = 1 - P(A)$.

$\boxed{\text{Proof of } P(A) \le P(B) \text{ if } A \subset B}$



Note that $B = A \cup (B \cap \bar{A})$

The shaded portion in the figure

Also, $A \cap (B \cap \bar{A}) = \phi$

Hence $P(B) = P(A) + P(B \cap \bar{A})$

$\ge P(A) \quad (\because P(B \cap \bar{A}) \ge 0)$