LECTURE – 24

Agenda: ① The Beta distribution
② Moment generating function

THE BETA DISTRIBUTION

A random variable X is said to have a Beta (α, β) distribution, if

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} & \text{for } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Here $\alpha > 0$, $\beta > 0$ are parameters of the distribution. Note that the above density is supported on the interval $[0, 1]$. This random variable is often used as a good model for quantities which are proportions. For example, proportion of diseased people in a population, proportion of defective items in a collection, etc.

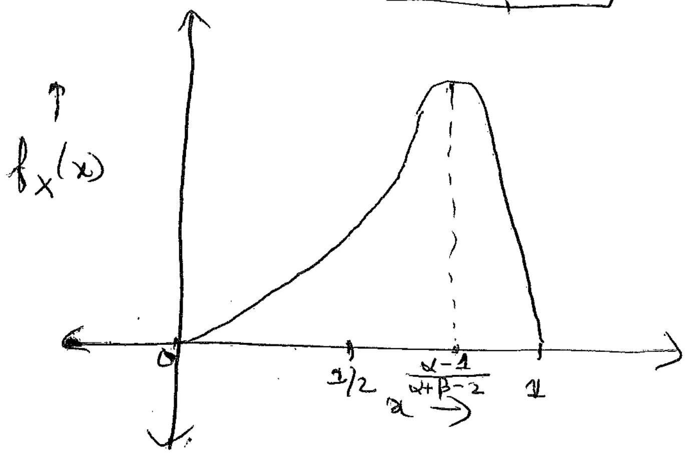
Also, if $\alpha = 1$, $\beta = 1$, then

$$f_X(x) = \begin{cases} \dfrac{\Gamma(2)\, x^{1-1}(1-x)^{1-1}}{\Gamma(1)\,\Gamma(1)} = 1 & \text{if } x \in [0,1], \\ 0 & \text{otherwise} \end{cases}$$

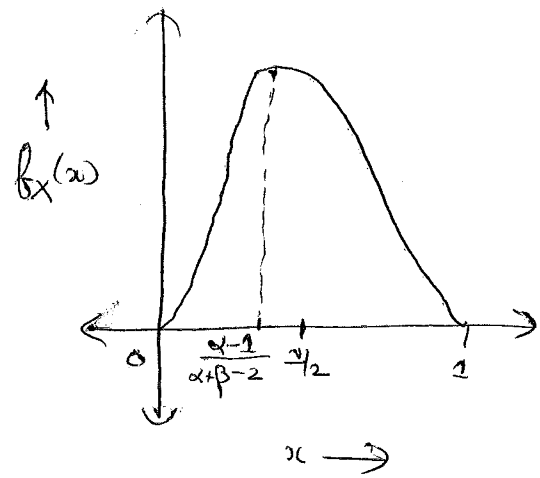
This is precisely the Uniform [0, 1] density. Hence, the Uniform [0, 1] density is a special case of the class of Beta densities.

We now examine the shape of the Beta density for some important ranges of α and β.

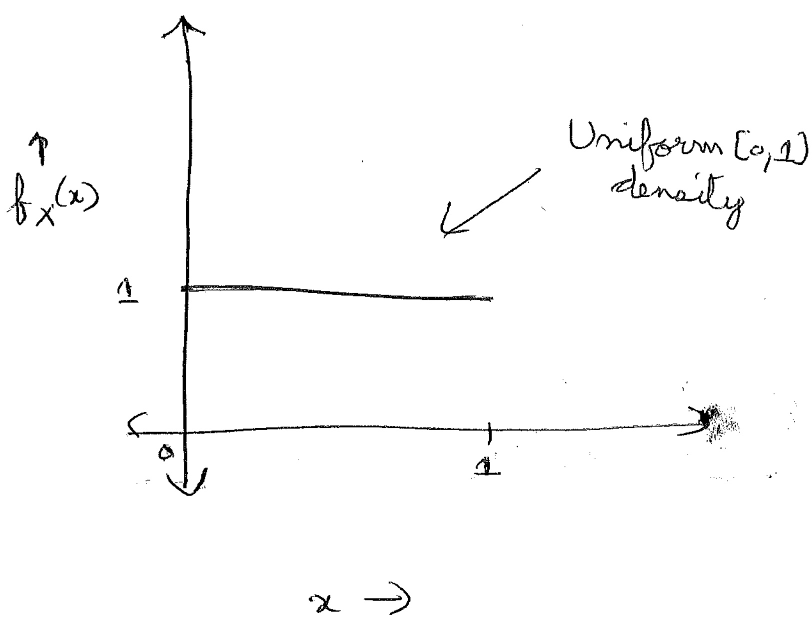
$\boxed{\alpha > \beta > 1}$



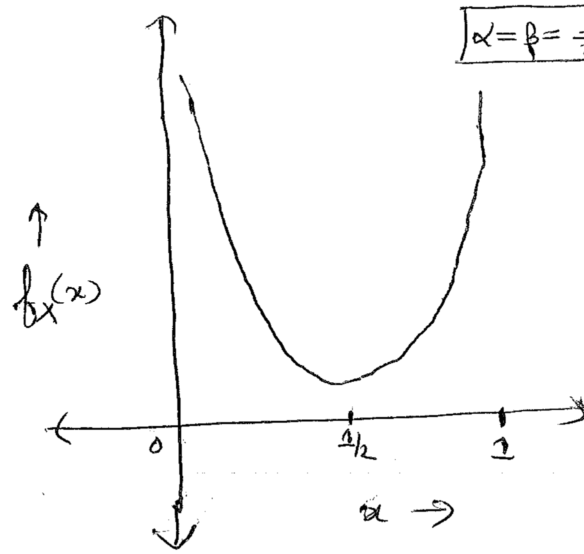
$\boxed{\beta > \alpha > 1}$



$\boxed{\alpha = \beta = 1}$



$\boxed{\alpha = \beta = \frac{1}{2}}$



We now proceed to deriving $E[X]$.

MATHEMATICAL IDENTITY: $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

for every $\alpha, \beta > 0$

$$E[X] = \int_0^1 \frac{x\, x^{\alpha-1}(1-x)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}$$ ($\because$ Use the above mathematical identity with $\alpha+1$ and β)

$$= \frac{\alpha}{\alpha+\beta}$$ $\left(\because \begin{array}{l}\Gamma(\alpha+1) = \alpha\Gamma(\alpha)\\ \Gamma(\alpha+\beta+1) = (\alpha+\beta)\Gamma(\alpha+\beta)\end{array}\right)$

Using similar arguments, it can be shown that

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

EXAMPLE: Suppose the proportion of ~~diseased people~~ time X that a sheet metal stamping machine is down for maintenance or repairs has a beta distribution with $\alpha = 1$ and $\beta = 2$. The cost (in \$100s) of this downtime in lost production and repair expenses is given by

$$C = 10 + 20X + 4X^2.$$

Find the mean of C.

Note that

$$E[C] = 10 + 20E[X] + 4E[X^2]$$
$$= 10 + 20E[X] + 4\left(V(X) + (E[X])^2\right)$$
$$= 10 + 20E[X] + 4(E[X])^2 + 4V(X).$$

Since $E[X] = \frac{\alpha}{\alpha+\beta} = \frac{1}{1+2} = \frac{1}{3}$, and

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2}{3\times3\times4} = \frac{1}{18},$$

it follows that

$$E[C] = 10 + \frac{20}{3} + \frac{4}{9} + \frac{4}{18}$$
$$= \frac{300+8+4}{18} = \frac{312}{18} = \frac{52}{3}.$$

MOMENT GENERATING FUNCTION

Given a random variable, X, many of its properties can be understood by looking at the moment generating function, defined by

$$\underbrace{M_X(t)}_{\boxed{MGF}} \triangleq E[e^{tX}] \quad \text{for every } t \in \mathbb{R}.$$

Note that the MGF can possibly be infinite for some or all values of t. We will examine this in more detail when we derive MGF for a variety of random variables.

The MGF gets its name from the fact that the moments of the random variable X, namely, $E[X]$, $E[X^2]$, $E[X^3]$, - - - - -
(In general, $E[X^k]$ is called the k-th moment of X)
can be obtained by differentiating this function. In particular,

$$E[X] = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$E[X^2] = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0},$$

and

$$E[X^k] = \frac{d^k}{dt^k} M_X(t) \Big|_{t=0}$$

for any positive integer k.