# LECTURE - 27

Agenda:

1. Joint mass functions of discrete random variables
2. Examples

## JOINT MASS FUNCTIONS OF DISCRETE RANDOM VARIABLES

In many applications, it is important to study the joint behaviour of two or more random variables. With this in mind, we define the notion of the joint mass function of two discrete random variables $X$ and $Y$.

**Definition:** The joint mass function $p_{X,Y}$ of two discrete random variables $X$ and $Y$ is defined as

$$p_{X,Y}(x,y) \triangleq P(X=x, Y=y)$$

for every $x \in \text{Range}(X)$ and $y \in \text{Range}(Y)$

**Example:** On April 15, 1912, the ocean liner Titanic sank. Of the 2201 people on the Titanic, 1490 perished. The question as to whether passenger class was related to survival has been discussed extensively.

Consider picking a passenger randomly from Titanic.
Let

$$X = \begin{cases} 0 & \text{if passenger survived} \\ 1 & \text{if passenger did not survive} \end{cases}$$

and

$$Y = \begin{cases} 1 & \text{if passenger was first class} \\ 2 & \text{if passenger was second class} \\ 3 & \text{if passenger was third class} \\ 4 & \text{if passenger was crew member} \end{cases}$$

Then, based on data, the joint mass function of X and Y can be summarized as follows.

| Y \ X | 0 | 1 |
|---|---|---|
| 1 | 0.09 | 0.06 |
| 2 | 0.05 | 0.08 |
| 3 | 0.08 | 0.24 |
| 4 | 0.10 | 0.30 |

Hence, for example,

$$P(X=0, Y=1) = 0.09$$
$$P(X=0, Y=2) = 0.05$$
$$P(X=1, Y=1) = 0.06$$
$$P(X=1, Y=2) = 0.08$$
$$P(X=0, Y=3) = 0.08$$

and so on.

If we know the joint mass function of X and Y, the marginal mass functions of X and Y can be easily worked out.

$$\boxed{\begin{aligned} p_X(x) &= P(X=x) = \sum_{y \in \text{Range}(Y)} P(X=x, Y=y) \\ p_Y(y) &= P(Y=y) = \sum_{x \in \text{Range}(X)} P(X=x, Y=y) \end{aligned}}$$

In the titanic example,

$$p_X(0) = P(X=0) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=0, Y=4)$$

$$= 0.32$$

$$p_X(1) = 1 - p_X(0) = 0.68$$

$$p_Y(1) = P(X=0, Y=1) + P(X=1, Y=1) = 0.15$$

$$p_Y(2) = P(X=0, Y=2) + P(X=1, Y=2) = 0.13$$

$$p_Y(3) = P(X=0, Y=3) + P(X=1, Y=3) = 0.32$$

$$p_Y(4) = P(X=0, Y=4) + P(X=1, Y=4) = 0.40$$

Finally knowing the joint mass function allows us to understand the conditional behaviour of one random variable given the other.

The conditional mass function of $Y$ given $X = x$ is defined as

$$p_{Y|X=x}(y) \triangleq P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

for every $y \in \text{Range}(Y)$

The conditional mass function of $Y$ given $X = x$, describes the behavior of $Y$ in a conditional world where $X = x$. One can similarly define the conditional mass function of $X$ given $Y = y$ as

$$p_{X|Y=y}(x) \triangleq P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

for every $x \in \text{Range}(X)$

Going back to the titanic example,

$$p_{X|Y=1}(0) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{9}{15}, \quad p_{X|Y=1}(1) = \frac{6}{15}$$

$$p_{X|Y=2}(0) = \frac{5}{13}, \quad p_{X|Y=2}(1) = \frac{8}{13}$$

$$p_{X|Y=3}(0) = \frac{8}{32}, \quad p_{X|Y=3}(1) = \frac{24}{32}$$

$$p_{X|Y=4}(0) = \frac{1}{4}, \quad p_{X|Y=4}(1) = \frac{3}{4}$$

Hence, the chance of survival seems to be the highest for first class passengers ($P(X=0 \mid Y=1) = \frac{9}{15}$) and lowest for third class passengers and crew members ($P(X=0 \mid Y=3) = P(X=0 \mid Y=4) = \frac{1}{4}$).