LECTURE (27)

Agenda:

(1) Joint distribution functions of continuous random variables

(2) Examples

JOINT DISTRIBUTION FUNCTIONS OF CONTINUOUS RANDOM VARIABLES

We previously studied the joint probability mass functions for jointly describing the probability behaviour of two discrete random variables. Today, we repeat the same exercise for continuous random variables.

Let us recollect that a random variable X is said to be continuous, if it has a density function f_X, such that

(i) $f_X(x) \geq 0$ for every $x \in \mathbb{R}$, $f_X(x) = 0$ for $x \notin \mathcal{X}$ $= \text{Range}(X)$

(ii) $P(a \leq X \leq b) = \underbrace{\int_a^b f_X(x)\,dx}_{}$.

Area under the curve f_X from a to b.

Suppose we have two continuous random variables X and Y. The joint probability behaviour of X and Y is described by the "<u>joint probability density function</u>" $f_{X,Y}$ which has the following properties.

(i) $f_{X,Y}(x,y) \geq 0$ for every $(x,y) \in \mathbb{R}^2$

(ii) $$P(a \leq X \leq b,\ c \leq Y \leq d) = \underbrace{\int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy}_{\substack{\text{Volume under surface} \\ f_{X,Y} \text{ from } [c,d]\times[a,b].}}$$

It follows that the "<u>joint probability distribution function</u>" $F_{X,Y}$ is given by

$$F_{X,Y}(a,b) = P(X \leq a,\ Y \leq b) = \underbrace{\int_{-\infty}^{b} \int_{-\infty}^{a} f_{X,Y}(x,y)\,dx\,dy}_{\substack{\text{Volume under} \\ \text{surface } f_{X,Y} \text{ from} \\ [-\infty,b]\times(-\infty,a]}}$$

<u>Example:</u> A certain process for producing an industrial chemical yields a product that contains two main types of impurities. Let X denote the proportion of impurities of Type I and Y denotes the proportion of impurities of Type II. Suppose that the joint density of X and Y can be adequately modeled by the following function.

$$f_{X,Y}(x,y) = \begin{cases} 2(1-x) & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute $P(0 \leq X \leq 0.5, 0.4 \leq Y \leq 0.7)$

$$P(0 \leq X \leq 0.5,\ 0.4 \leq Y \leq 0.7)$$

$$= \int_{0.4}^{0.7} \int_{0}^{0.5} 2(1-x)\,dx\,dy$$

$$= \int_{0.4}^{0.7} \left[-(1-x)^2\right]_0^{0.5} dy$$

$$= \int_{0.4}^{0.7} 0.75\,dy$$

$$= (0.75) \int_{0.4}^{0.7} dy$$

$$= (0.75) \times (0.3)$$

$$= 0.225.$$

<u>RESULT:</u> If X and Y are continuous random variables with joint probability density function $f_{X,Y}$, then the individual or marginal density functions f_X and f_Y are given by the following:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{for every } x \in \mathbb{R},$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad \text{for every } y \in \mathbb{R}.$$

In the industrial production example considered previously, compute f_X and f_Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \begin{cases} \int_0^1 2(1-x)\,dy & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2(1-x)\,[y]_0^1 & 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2(1-x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \int_0^1 2(1-x)\,dx & 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose we are interested in the behaviour of the random variable X given that $Y = y$. To develop a framework for expressing this behaviour, we need the notion of "conditional probability density function".

DEFINITION: Let X and Y be continuous random variables with joint probability density function $f_{X,Y}$ and marginal densities f_X and f_Y. Then the conditional probability density function of X given $Y = y$ is defined by

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & y \in \mathcal{Y} = \text{Range}(Y), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the conditional probability density function of Y given $X = x$, is defined by

$$f_{Y|X=x} = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} & x \in \mathcal{X} = \text{Range}(X), \\ 0 & \text{otherwise.} \end{cases}$$

Compute the conditional probability density functions $f_{X|Y=y}$ and $f_{Y|X=x}$ for the industrial impurities example.

$$f_{X|Y=y} = \begin{cases} \dfrac{2(1-x)}{1} & 0 \le x \le 1, \ (0 \le y \le 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2(1-x) & 0 \le x \le 1, \ (0 \le y \le 1), \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{Y|X=x} = \begin{cases} \dfrac{2(1-x)}{2(1-x)} & 0 \le y \le 1, \ (0 \le x \le 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & 0 \le y \le 1 \quad (0 \le x \le 1), \\ 0 & \text{otherwise.} \end{cases}$$

We see that $f_{X|Y=y}$ is the same as f_X, and $f_{Y|X=x}$ is the same as f_Y. This means that the random variables X and Y behave independently, since fixing one to any value (in the appropriate range) does not affect the probability behaviour of the other. We consider this more formally in the next lecture.