LECTURE - 29

__Agenda:__

1. Independent Random Variables
2. Expected values of functions of random variables

INDEPENDENT RANDOM VARIABLES

Let us recollect the notion of "independent events". We say that events A and B are independent if

$$P(A \cap B) = P(A)P(B) \quad \text{or} \quad P(A \mid B) = P(A).$$

The way we understand it intuitively is that even if we are given the information that B has occured, that does not change the probability of A. The same notion can be generalized to random variables. Let us consider the case of discrete random variables.

__Definition:__ Let X, Y be discrete random variables. Then, X and Y are said to be independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

__for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$.__

Note that
$$P(X = x, Y = y) = P(X = x)\,P(Y = y)$$

is the same as

$$P(X = x \mid Y = y) = P(X = x) \text{ or } P(Y = y \mid X = x) = P(Y = y)$$

Hence, saying that X and Y are independent, means that even if we are given the information that $Y = y$, that does not change the probability behaviour of X. Similarly, even if we are given the information that $X = x$, that does not change the probability behaviour of Y.

Let us now turn our attention to continuous random variables.

Definition: Let X, Y be continuous random variables. Then, X and Y are said to be independent if

$$f_{X,Y}(x, y) = f_X(x)\, f_X(y) \quad \text{for every } x \in \mathbb{R}, y \in \mathbb{R}.$$

Note that
$$f_{X,Y}(x, y) = f_X(x)\, f_X(y)$$

is the same as.

[redacted]

$$f_{X|Y=y}(x) = f_X(x) \quad \text{or} \quad f_{Y|X=x}(y) = f_Y(y).$$

Hence, saying that X and Y are independent, implies that the probability behaviour of X is unaffected by information about Y, and the probability behaviour of Y is unaffected by information about X.

<u>Example:</u> A bus arrives at a bus stop at a randomly selected time within a 1- hour period. A passenger arrives at the bus stop at a randomly selected time within the same hour. The passenger is willing to wait for the bus for up to $\frac{1}{4}$ of an hour. What is the probability that the passenger will catch the bus?

Let X = Bus arrival time and Y = Passenger arrival time

X is uniform on $[0,1]$ and Y is uniform on $[0,1]$. Hence,

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 1 & \text{if } 0 \le y \le \\ 0 & \text{otherwis} \end{cases}$$

Since X and Y are independent, the joint probability density of (X, Y) is the product of the individual or marginal probability densities of X and Y.

Hence, the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the random variables,

$P($ Passenger catches the bus$)$

$= P(Y \le X \le Y + \frac{1}{4})$

$= P((X,Y) \in A)$

where $A = \{(x,y) \in \mathbb{R}^2 : y \le x \le y + \frac{1}{4}\}$

<u>RESULT:</u> If X and Y are continuous random variables with joint density $f_{X,Y}$, then

$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy \quad \text{for any}$$

subset A of $\mathbb{R}^2$.

Hence, in our particular example,

$$P\big((X,Y) \in A\big) = \iint_{\{(x,y)\in\mathbb{R}^2 :\ y \le x \le y+\frac{1}{4}\}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{y}^{y+\frac{1}{4}} f_{X,Y}(x,y)\,dx \right) dy$$

$$= \int_{0}^{1} \left(\int_{y}^{y+\frac{1}{4}} f_{X,Y}(x,y)\,dx \right) dy$$

(Since $f_{X,Y}(x,y) = 0$ if $y < 0$ or $y > 1$)

$$= \int_{0}^{3/4} \left(\int_{y}^{y+\frac{1}{4}} 1\,dx \right) dy + \int_{3/4}^{1} \left(\int_{y}^{1} 1\,dx \right) dy$$

(Since $f_{X,Y}(x,y) = 0$ if $x < 0$ or $x > 1$,

and if $\frac{3}{4} < y$, then the interval $(y, y+\frac{1}{4})$

crosses the point 1)

$$= \int_{0}^{3/4} \frac{1}{4}\,dy + \int_{3/4}^{1} (1-y)\,dy$$

$$= \frac{1}{4} \times \frac{3}{4} + \left[-\frac{(1-y)^2}{2} \right]_{3/4}^{1}$$

$$= \frac{3}{16} + \frac{1}{32}$$

$$= \frac{7}{32}.$$

Hence the probability that the passenger catches the bus is $\frac{7}{32}$.

EXPECTED VALUES OF FUNCTIONS OF RANDOM VARIBLES

If X, Y are discrete random variables with joint probability mass function $p_{X,Y}(x,y)$, then for any function $g: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ (where $\mathcal{X} = \text{Range}(X)$ and $\mathcal{Y} = \text{Range}(Y)$),

$$E[g(X,Y)] = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} g(x,y) p_{X,Y}(x,y)$$

If X, Y are continuous random variables with joint probability density function $f_{X,Y}(x,y)$, then for any function $g: \mathbb{R}^2 \to \mathbb{R}$,

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy.$$