LECTURE – 30

1. Probabilities involving two random variables
2. Expectations involving two random variables

We learnt in the last lecture that if (X,Y) have joint density $f_{X,Y}$, then for any $A \subseteq \mathbb{R}^2$,

$$P\left((X,Y) \in A \right) = \iint_A f_{X,Y}(x,y)\,dx\,dy.$$

Also, typically in this course, the joint density of (X,Y) will have the form

$$f_{X,Y}(x,y) = \begin{cases} h(x,y) > 0 & \text{if } (x,y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

We call R as the range of the joint density $f_{X,Y}$. We now provide a general procedure for evaluating $P\left((X,Y) \in A \right)$.

(1) Note that $P((X,Y)\in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy = \iint_{A\cap R} h(x,y)\,dx\,dy$

Hence the job is now to evaluate $\iint_{A \cap R} h(x,y)\,dx\,dy$.

(2) Figure out the range of x values that can be taken in $A \cap R$.

- Call this set $\mathcal{X}_A$. For every $x \in \mathcal{X}_A$, find the projection set

$$A_x = \{ y : (x, y) \in A \cap R \}.$$

(3) Basic theory of calculus tells us that

$$\iint_{A \cap R} h(x, y)\, dy\, dx = \int_{\mathcal{X}_A} \int_{A_x} h(x, y)\, dy\, dx$$

In most examples, both $\mathcal{X}_A$ and A_x will be either intervals or union of intervals. Hence, simplifying the integral further involves standard integration techniques.

__EXAMPLE:__ Suppose that the joint density of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} e^{-x} & \text{if } 0 \le y \le x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X < 2, Y > 1)$.

Always convert the probability into the form $P((X,Y) \in A)$.
Here,

$$P(X<2, Y<1) = P\left((X,Y) \in A\right), \text{ where}$$

$$A = \{ (x,y) : x<2, y<1\}.$$

① Note that the range of the density function is

$$R = \{ (x,y) : 0 \le y \le x < \infty\}.$$

Hence $A \cap R = \{ (x,y) : x<2, y<1, 0 \le y \le x < \infty \}$.

It follows that

$$P((X,Y) \in A) = \iint_{A \cap R} e^{-x} \, dy \, dx$$

② Clearly, the range of x values that can be taken in $A \cap R$ is $\mathcal{X}_A = [0,2]$.

Fix $x \in \mathcal{X}_A$. Then

$$\begin{aligned} A_x &= \{ y : (x,y) \in A \cap R \} \\ &= \{ y : y<1, 0 \le y \le x < \infty \} \\ &= \{ y : 0 \le y \le \min(1,x) \} \end{aligned}$$

③ It follows that

$$P((X,Y) \in A) = \int_0^2 \int_0^{\min(1,x)} e^{-x} \, dy \, dx$$

$$= \int_0^2 \min(1,x)\, e^{-x}\, dx$$

$$= \int_0^1 x e^{-x}\, dx + \int_1^2 e^{-x}\, dx$$

$$= \int_0^1 \underbrace{x}_{\mathrm{I}}\, \underbrace{e^{-x}}_{\mathrm{II}}\, dx + \left[-e^{-x}\right]_1^2$$

(Integration by parts)

$$= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x} + (e^{-1} - e^{-2})$$

$$= e^{-0} - e^{-1} - e^{-2}$$

$$= 1 - e^{-1} - e^{-2}.$$

Hence,

$$P(X<2, Y<1) = 1 - e^{-1} - e^{-2}.$$

EXPECTATIONS OF FUNCTIONS OF TWO CONTINUOUS RANDOM VARIABLES

If (X,Y) have joint density $f_{X,Y}$, then

$$E[g(X,Y)] \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$$

Note again that if $f_{X,Y}(x,y) = \begin{cases} h(x,y) > 0 & \text{if } (x,y) \in A, \\ 0 & \text{otherwise.} \end{cases}$

then,

$$E[g(X,Y)] = \iint_R g(x,y)\, h(x,y)\, dy\, dx.$$

Hence, the same process as earlier needs to be repeated with R (instead of $A \cap R$) and $g(x,y)\,h(x,y)$ instead of $h(x,y)$. For example, if $g(x,y) = y$, then using the same joint density as in the previous example,

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \iint_{\{(x,y):\, 0 \le y \le x < \infty\}} y e^{-x}\, dy\, dx$$

$$= \int_0^{\infty}\int_0^{x} y e^{-x}\, dy\, dx$$

$$= \int_0^{\infty} e^{-x} \left[\frac{y^2}{2}\right]_0^{x} dx$$

$$= \int_0^{\infty} \frac{x^2 e^{-x}}{2}\, dx$$

$$= \frac{\Gamma(3)}{2}$$

$$= \frac{2!}{2}$$

$$= 1.$$

You need not always use the order $\boxed{dy\,dx}$. If it is more convenient, you can also use $\boxed{dx\,dy}$. For example, if $g(x,y)=x$, then

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\{(x,y):\, 0\le y\le x<\infty\}} x e^{-x}\,dx\,dy$$

$$= \int_0^{\infty}\left(\int_y^{\infty} \underbrace{x}_{I}\,\underbrace{e^{-x}}_{II}\,dx\right)dy \qquad \text{(Integration by parts)}$$

$$= \int_0^{\infty}\left(\left[-x e^{-x}\right]_y^{\infty} + \int_y^{\infty} e^{-x}\,dx\right)dy$$

$$= \int_0^{\infty}\left(y e^{-y} + e^{-y}\right)dy$$

$$= \Gamma(2) + \Gamma(1)$$

$$= 1! + 0!$$

$$= 2.$$