LECTURE - 37

Agenda:

1. Convergence in distribution
2. Central limit theorem

CONVERGENCE IN DISTRIBUTION

Consider the following situation:

We have an experiment which gives rise to n random variables say $X_1, X_2, \ldots, X_n$ which are independent and have the same probability distribution. We are interested in the probability behaviour of $X_1 + X_2 + \cdots + X_n$. In general, it may not always be possible to apply any of the methods to find the exact distribution of this random variable. However, if n is large enough, then we can approximately find the probability distribution of $X_1 + X_2 + \cdots + X_n$. To this end, we introduce the notion of convergence in distribution for random variables.

Definition: Let $X_1, X_2, X_3, \ldots\ldots$ be a sequence of random variables. We say that this sequence of random variables converges in distribution to the random variable X if

$$\lim_{n \to \infty} \underbrace{P(X_n \le x)}_{F_{X_n}(x)} = \underbrace{P(X \le x)}_{F_X(x)}$$

for every $x \in \mathbb{R}$ such that F_X is continuous at x.

As is clear from the definition, this means that if n is large enough, then the distribution of X_n is approximately the same as the distribution of X.

Here are some examples illustrating this concept in familiar situations.

(1) Let $X_1, X_2, X_3, \ldots\ldots\ldots$ be a sequence of random variables such that

$$X_n \text{ is Binomial } (n, p_n),$$

and $np_n \to \lambda$ as $n \to \infty$.

Then $X_1, X_2, X_3, \ldots\ldots\ldots$ converges in distribution to $X = \text{Poisson}(\lambda)$.

<u>TAKE AWAY:</u> If you have a Binomial random variable with parameters n and p such that n is fairly large and p is fairly small, then its distribution is very well approximated by a Poisson (np) random variable.

(2) Let $X_1, X_2, X_3, \ldots\ldots\ldots$ be a sequence of random variables such that

$$X_n \text{ is Poisson}(n).$$

Then $\frac{X_1 - 1}{\sqrt{1}}, \frac{X_2 - 2}{\sqrt{2}}, \frac{X_3 - 3}{\sqrt{3}}, \ldots\ldots\ldots$ converges in distribution to $X = N(0, 1)$

<u>TAKE AWAY:</u> If you have $Y = \text{Poisson}(\lambda)$ where λ is fairly large, then the distribution of

$$\frac{Y-\lambda}{\sqrt{\lambda}}$$

is very well approximated by a $N(0,1)$ random variable.

THE CENTRAL LIMIT THEOREM

We saw an example of a sequence of random variables which converges in distribution to a $N(0,1)$ random variable. This phenomenon is shared by a large class of random variables.

<u>RESULT:</u> Let $X_1, X_2, X_3, \ldots\ldots$ be independent random variables with the same distribution. Let $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ (which is finite)

Define

$$Y_n = \sqrt{n}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma}\right).$$

for each $n \geq 1$.

Then $Y_1, Y_2, Y_3, \ldots\ldots$ converges in distribution to a $N(0,1)$ random variable.

TAKE AWAY: If $X_1, X_2, \ldots, X_n$ are independent random variables with the same distribution, and $E(X_i) = \mu$, and $V(X_i) = \sigma^2$ with σ^2 finite, then, for fairly large n,

$$P\left(\sum_{i=1}^{n} X_i \le x \right) \approx \Phi\left(\frac{x - n\mu}{\sqrt{n}\,\sigma} \right)$$

Distribution function of the standard normal random variable.

RULE: $n \ge 30$ is large enough generally.

Example: A certain machine that is used to fill bottles with liquid has been observed over a long period, and the variance in the amounts of fill has been found to be approximately $\sigma^2 = 1$ ounce. The mean ounces of fill has been observed to be 10 ounces. Suppose 36 bottles are filled on a given day. Find the probability that total volume in these bottles is less than 370 ounces, by using the central limit theorem.

Let X_i = Amount in bottle i, $i = 1, 2, \ldots, 36$

$X_1, X_2, \ldots, X_{36}$ are independent and have the same distribution. Also, $E(X_i) = 10$ and $V(X_i) = 1$. We wish to evaluate $P\left(\sum_{i=1}^{36} X_i \le 370 \right)$.

Since $n = 36$ is large enough, we use the central limit theorem to obtain the approximation

$$P\left(\sum_{i=1}^{36} X_i \le 370\right) \approx \Phi\left(\frac{370 - 10 \times 36}{\sqrt{36} \times 1}\right)$$

$$= \Phi\left(\frac{10}{6}\right)$$

$$= \Phi(1.66)$$

$$\simeq 974.$$