SOLUTIONS TO EXAM 1

PROBLEM 1: (a)

$$E[\bar{Y}^2] = V(\bar{Y}) + (E[\bar{Y}])^2$$

$$= V\left(\frac{\sum Y_i}{n}\right) + \left(E\left[\frac{\sum Y_i}{n}\right]\right)^2$$

$$= \frac{\sum_{i=1}^{n} V(Y_i)}{n^2} + \left(\frac{\sum_{i=1}^{n} E[Y_i]}{n}\right)^2$$

($\because$ By independence)

$$= \frac{n\lambda}{n^2} + \left(\frac{n\lambda}{n}\right)^2$$

$$= \lambda^2 + \frac{\lambda}{n}.$$

(b)

$$E\left[\bar{Y}^2 - \frac{\bar{Y}}{n}\right] = \lambda^2 + \frac{\lambda}{n} - \frac{E[\bar{Y}]}{n} \quad (\because \text{By Part (a)})$$

$$= \lambda^2 + \frac{\lambda}{n} - \frac{\lambda}{n} \quad (\because E[\bar{Y}] = \lambda \text{ as shown in Part (a)})$$

$$= \lambda^2$$

Hence, $\bar{Y}^2 - \frac{\bar{Y}}{n}$ is an unbiased estimator of λ^2.

(c) $E\left[\bar{Y}^2 + 5\bar{Y} - \frac{\bar{Y}}{n}\right] = \lambda^2 + \frac{\lambda}{n} + 5\lambda - \frac{\lambda}{n} = \lambda^2 + 5\lambda.$

Hence $\bar{Y}^2 + 5\bar{Y} - \frac{\bar{Y}}{n}$ is an unbiased estimator of $\lambda^2 + 5\lambda$.

PROBLEM 2: (a) As discussed in class,

$$E[S^2] = \sigma^2 \text{ and } E[\bar{Y}] = \mu.$$

Hence, $E[S^2 + 2\bar{Y}] = E[S^2] + 2E[\bar{Y}] = \sigma^2 + 2\mu$, and $S^2 + 2\bar{Y}$ is an unbiased estimator of $\sigma^2 + 2\mu$.

(b) χ^2 distribution with $n-1$ degrees of freedom.

(c) $\frac{(n-1)}{\sigma^2} S^2 \sim \chi^2_{n-1} = \text{Gamma}\left(\alpha = \frac{n-1}{2}, \beta = 2\right)$

Hence $E[S^8] = E[(S^2)^4] = \frac{\sigma^8}{(n-1)^4} E\left[\left(\frac{(n-1)S^2}{\sigma^2}\right)^4\right]$

$$= \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\left(\frac{n-1}{2}+2\right)\left(\frac{n-1}{2}+3\right) 2^3 \frac{\sigma^8}{(n-1)^4}$$

$$= \frac{(n-1)(n+1)(n+3)(n+5)\sigma^8}{2(n-1)^4}$$

$$= \frac{(n+1)(n+3)(n+5)\sigma^8}{2(n-1)^3}.$$

PROBLEM 3: (a) $P(0.5 \le W \le 175) = 0.95$

$$\Rightarrow P\left(0.5 \le \frac{2}{\theta}\sum_{i=1}^{100} Y_i \le 175\right) = 0.95$$

$$\Rightarrow P\left(\frac{2\sum_{i=1}^{100} Y_i}{175} \le \theta \le 4\sum_{i=1}^{100} Y_i\right) = 0.95.$$

Hence $\left[\frac{2\sum_{i=1}^{100}}{175}, 4\sum_{i=1}^{100} Y_i\right]$ is a 95% confidence interval for θ.

$\boxed{(b)}$ If $\sum_{i=1}^{100} Y_i = 200$, then the 95% confidence interval is given by

$$\left[\frac{2 \times 200}{175}, 4 \times 200\right] = \left[\frac{400}{175}, 800\right]$$

$$= [2.29, 800].$$

Problem 4: $\boxed{(a)}$ $P(-1.98 \leq W \leq 1.98) = 0.95$

$$\Rightarrow P\left(-1.98 \leq \frac{\sqrt{200}\,(\overline{Y}-\mu)}{S} \leq 1.98\right) = 0.95$$

$$\Rightarrow P\left(\overline{Y} - \frac{1.98S}{\sqrt{200}} \leq \mu \leq \overline{Y} + \frac{1.98S}{\sqrt{200}}\right) = 0.95.$$

Hence the 95% confidence interval for μ is given by

$$\left[\overline{Y} - \frac{1.98S}{\sqrt{200}}, \overline{Y} + \frac{1.98S}{\sqrt{200}}\right].$$

$\boxed{(b)}$ If $\sum_{i=1}^{200} Y_i = 40$, $\overline{Y} = \frac{40}{200} = \frac{1}{5}$, and $S = \sqrt{S^2} = 10$, then the 95% confidence interval for μ is given by

$$\left[0.2 - \frac{19.8}{\sqrt{200}}, 0.2 + \frac{19.8}{\sqrt{200}}\right] = [-1.2, 1.6].$$