

## LECTURE 3

### Agenda:

- ① Some examples to understand the formal definition of probability
- ② Fundamental principle of counting
- ③ Evaluating probabilities using permutations.

### EXAMPLES

Random Experiment: Choose a person from 4 persons at random, with no preference to any person.

- What is the sample space?

Since there are 4 possible outcomes, the sample space is  $S = \{1, 2, 3, 4\}$ .

- What is the set of all possible events?

Recall that  $\mathcal{A}$ , which denotes the set of all possible events, is essentially the collection of all possible subsets of  $S$ . Hence,

$$\mathcal{A} = \left\{ \begin{array}{l} \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \\ \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \\ \{1, 2, 3, 4\} \end{array} \right\}$$

- How to provide a probability assignment  $P$ , which reflects our belief in how the random experiment is conducted, and also satisfies the 3 axioms?

SOLUTION: Based on your belief of how the random experiment is conducted, assign a chance to EACH POINT IN THE SAMPLE SPACE  $\mathcal{S}$ , such that these numbers add up to 1. For this experiment, since there is no preference for ~~any~~ <sup>any</sup> person,

$$P(\{1\}) = \frac{1}{4}, P(\{2\}) = \frac{1}{4}, P(\{3\}) = \frac{1}{4}, P(\{4\}) = \frac{1}{4}$$

DEFINE THE PROBABILITY OF ANY EVENT AS THE SUM OF PROBABILITIES OF THE SAMPLE POINTS IN THE GIVEN EVENT.

Hence, for  $A =$  Event that Person 1 or Person 2 is chosen,

$$P(A) = P(\{1,2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

This procedure guarantees that the probability assignment ~~is~~ satisfies the 3 axioms, for discrete sample spaces.

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Do we always need to go through this procedure for calculating probabilities of events? No.

We can often use counting rules to get around the situation.

COUNTING RULE #1 Suppose we are performing an experiment where all outcomes are equally likely, hence we assign the same probability to every sample point (say there are  $N$  sample points).

Suppose the event of interest, say  $A$ , consists of  $n_A$  sample points. Then,

$$P(A) = \frac{n_A}{N} = \frac{\# \text{ sample points in } A}{\# \text{ total sample points}}$$

### THE FUNDAMENTAL PRINCIPLE OF COUNTING

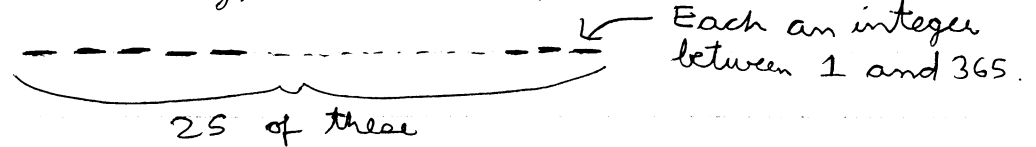
Suppose that an experiment consists of two successive tasks. The first task can result in  $n_1$  outcomes and for each such outcome, the second task can result in  $n_2$  outcomes. Then the total number of outcomes of the experiment is  $n_1 n_2$ .

Example: Birthday problem

Random experiment: Choose 25 people. ~~365~~

Task: Evaluate the probability of the event that there is atleast 1 match in the 25 birthdays.

- Sample space is the collection of all possible 25-sequences of birthdays. A typical sample point looks like



Hence, number of sample points  $N = (365)^{25}$  (ignoring leap years)

- Assuming all birthdays are equally likely, each sample point has probability  $\frac{1}{(365)^{25}}$
- How many different ways are there for 25 people to have no common birthdays?

The answer is  $365 \cdot 364 \cdot 363 \cdot \dots \cdot 341 = \frac{365!}{340!}$

Hence,  $n_A = (365)^{25} - \frac{365!}{340!}$

$$\text{Hence, } P(A) = \frac{n_A}{N} = \frac{(365)^{25} - \frac{365!}{240!}}{(365)^{25}} = 0.5687$$

## PERMUTATIONS

Here are some identities which we have been using informally upto now, and which are quite helpful in counting principles for determining probabilities of events.

- What is the number of ways of choosing  $r$  objects with replacement from  $n$  objects (order is important)?

There are  $n$  different ways to choose each object. Hence the answer is  $n \times n \dots \times n = n^r$

- What is the number of ways to choosing  $r$  objects without replacement from  $n$  objects (order is important)?

The first slot can be filled in  $n$  ways, the second slot in  $(n-1)$  ways, the third slot in  $(n-2)$  ways and so on. Hence, the answer is.

$$n(n-1)(n-2) \dots (n-r+2)(n-r+1) \blacktriangle$$