LECTURE 3

Agenda:

1. Some examples to understand the formal definition of probability
2. Fundamental principle of counting
3. Evaluating probabilities using permutations.

EXAMPLES

Random Experiment: Choose a person from 4 persons at random, with no preference to any person.

- What is the sample space?
  Since there are 4 possible outcomes, the sample space is $S = \{1, 2, 3, 4\}$.

- What is the set of all possible events?
  Recall that $\mathcal{A}$, which denotes the set of all possible events, is essentially the collection of all possible subsets of S. Hence,

$$\mathcal{A} = \left\{ \begin{array}{l} \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \\ \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \\ \{1,2,3,4\} \end{array} \right\}$$

- How to provide a probability assignment P, which reflects our belief in how the random experiment is conducted, and also satisfies the 3 axioms?

SOLUTION: Based on your belief of how the random experiment is conducted, assign a chance to EACH POINT IN THE SAMPLE SPACE S, such that these numbers add up to 1. For this experiment, since there is no preference for any person,

$$P(\{1\}) = \frac{1}{4}, \quad P(\{2\}) = \frac{1}{4}, \quad P(\{3\}) = \frac{1}{4}, \quad P(\{4\}) = \frac{1}{4}$$

DEFINE THE PROBABILITY OF ANY EVENT AS THE SUM OF PROBABILITIES OF THE SAMPLE POINTS IN THE GIVEN EVENT.

Hence, for A = Event that Person 1 or Person 2 is chosen,

$$P(A) = P(\{1,2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

This procedure guarantees that the probability assignment satisfies the 3 axioms, for discrete sample spaces.

<u>Random Experiment:</u> There are 3 mailboxes. Three people come and put letters at random (with no preference to any of the three mailboxes).

<u>Task:</u> Compute the probability of the event that each mailbox is chosen once.

- Each outcome is a sequence of three numbers, each number denoting the mailbox chosen by the respective person. Hence,

$$S = \{123, 122, 131, \text{---------}\}$$

$$\boxed{27 \text{ points in total}}$$

- Since the mailboxes have no special preference by any person, all outcomes are equally likely.

  Hence we assign $P(\{s\}) = \frac{1}{27}$ for all $s \in S$.

- A = Event that each mailbox is chosen once.

$\Rightarrow A = \{123, 132, 213, 231, 312, 321\}$

Hence, $P(A) = P(\{123\}) + P(\{132\}) + P(\{213\}) + P(\{231\}) + P(\{312\}) + P(\{321\})$

$$= \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}$$

$$= \frac{6}{27}$$

Do we always need to go through this procedure for calculating probabilities of events? No. We can often use counting rules to get around the situation.

COUNTING RULE #1 Suppose we are performing an experiment where all outcomes are equally likely, hence we assign the same probability to every sample point (say there are N sample points).

Suppose the event of interest, say A, consists of n_A sample points. Then,

$$P(A) = \frac{n_A}{N} = \frac{\#\text{ sample points in } A}{\#\text{ total sample points}}$$

THE FUNDAMENTAL PRINCIPLE OF COUNTING

Suppose that an experiment consists of two successive tasks. The first task can result in n_1 outcomes and for each such outcome, the second task can result in n_2 outcomes. Then the total number of outcomes of the experiment is $n_1 n_2$.

Example: Birthday problem

Random experiment: Choose 25 people.

Task: Evaluate the probability of the event that there is atleast 1 match in the 25 birthdays.

- Sample space is the collection of all possible 25-sequences of birthdays. A typical sample point looks like

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \cdots\cdots\ \_\ \_\ \_}_{25 \text{ of these}} \quad \leftarrow \text{Each an integer between 1 and 365.}$$

Hence, number of sample points $N = (365)^{25}$ (ignoring leap years)

- Assuming all birthdays are eqrally likely, each sample point has probability $\frac{1}{(365)^{25}}$

- How many different ways are there for 25 people to have no common birthdays?

The answer is $365 \cdot 364 \cdot 363 \cdot \, \text{-----} \, \cdot 341 = \frac{365!}{340!}$

Hence, $n_A = (365)^{25} - \frac{365!}{340!}$

Hence, $P(A) = \frac{n_A}{N} = \frac{(365)^{25} - \frac{365!}{340!}}{(365)^{25}} = 0.5687$

PERMUTATIONS

Here are some identities which we have been using informally uptil now, and which are quite helpful in counting principles for determining probabilities of events.

- What is the number of ways of choosing r objects <u>with replacement</u> from n objects (order is important)?

  There are n different ways to choose each object. Hence the answer is $n \times n \times - - - \times n = n^r$

- What is the number of ways to choosing r objects <u>without replacement</u> from n objects (order is important)?

  The first slot can be filled in n ways, the second slot in $(n-1)$ ways, the third slot in $(n-2)$ ways and so on. Hence, the answer is.

  $$n(n-1)(n-2) - - - (n-r+2)(n-r+1) \blacktriangle.$$