LECTURE 4

Agenda:

① Counting rules.
② Examples

COUNTING RULES

Recall from Lecture 3 that

- # of ways of choosing r objects from n distinct objects with replacement (order is important) is n^r
- # of ways of choosing r objects from n distinct objects without replacement (order is important) is P_r^n, where

$$P_r^n = \frac{n!}{(n-r)!} . \quad \text{(PERMUTATIONS)}$$

RESULT: # of ways of choosing r objects from n distinct objects without replacement (order is not important) is

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{((COMBINATIONS)}$$

Proof: Note that now we are only interested in which objects were chosen and not the order in which they are chosen. Since every collection of r objects can be ordered in $r!$ ways, it follows that

1 combination corresponds to $r!$ permutations. Hence, the number of ways of choosing r objects without replacement from n distinct objects (order is not important) is $\frac{1}{r!} P_r^n = \frac{n!}{(n-r)!\, r!}$.

RESULT: # of ways of partitioning n distinct objects into k groups containing $n_1, n_2, ---, n_k$ objects, where $\sum_{i=1}^{k} n_i = n$, is

$$\frac{n!}{n_1!\, n_2! --- n_k!}$$

PROOF: The n_1 objects for the first group can be chosen in $\binom{n}{n_1}$ ways. The n_2 objects for the second group can be chosen in $\binom{n-n_1}{n_2}$ ways, ------,

the n_k objects for the k^{th} group can be chosen in $\binom{n-n_1-n_2 ---- n_{k-2}-n_{k-1}}{n_k}$ ways. Hence, the number of ways of partitioning n distinct objects into k groups containing $n_1, n_2, ---, n_k$ objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2} --- \binom{n-n_1-n_2 ---- n_{k-2}-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\, n_2!\, n_3! --- n_{k-1}!\, n_k!}$$

RESULT: # of ways of choosing r objects from n distinct objects with replacement (order is not important) is $\binom{n+r-1}{r}$.

Proof: Since order is not important, every way of choosing r objects uniquely corresponds to a vector of the form $(l_1, l_2, ---, l_n)$ where l_i represents the number of times the i^{th} object is chosen in the r draws. Let us represent the n-tuple $(l_1, l_2, ---, l_n)$ as follows

$$\underbrace{00---0}_{l_1 \text{ times}} \Big| \underbrace{00---0}_{l_2 \text{ times}} \Big| ------- \Big| \underbrace{00---0}_{l_n \text{ times}}$$

Note that $\sum_{i=1}^{n} l_i = r$. It follows that the number of vectors $(l_1, l_2, ---, l_n)$ with $l_i \geq 0$ and $\sum_{i=1}^{n} l_i = r$ is same as the number of ways of arranging $n-1$ "$|$" symbols and r "0" symbols. But the number of such ways is $\frac{(n-1+r)!}{(n-1)!\ r!}$ (we divide by $(n-1)!$ and $r!$ because the $(n-1)$ "$|$" symbols are indistinguishable and the r "0" symbols are indistinguishable. Hence the number of ways of choosing r objects with replacement from n distinct objects (order is not important) is $\binom{n+r-1}{r}$.

EXAMPLES

Example 1: Suppose a company manufactures 50 different machines out of which 4 are defective. A customer buys 3 machines. Find the probability that all three machines are defective, assuming no preference for any machine

Solution: Let us number the machines from 1 to 50. A typical sample point looks like

$$\underbrace{\_\_\ \_\_\ \_\_}_{\text{3 different integers between 1 to 50}}\ .$$

Hence, # of sample points $= 50 \times 49 \times 48 \; (= N)$.

A = Event that all machines are defective.

Hence, # of sample points in $A = 4 \times 3 \times 2 \; (= n_A)$.

Hence, $P(A) = \dfrac{n_A}{N} = \dfrac{4 \times 3 \times 2}{50 \times 49 \times 48} = 0 \cdot 0002$.

(Assuming no preference for any machine in the buy/sell process.)

Example 2: A company makes n orders. There are n distributors. There is no preference for a specific distributor. Find the probability that distributor 1 gets exactly k orders.

Solution: # of sample points $= M^n$.

(There are M distributors to choose from for each of n orders.)

A = Event that distributor 1 gets exactly k orders.

# of sample points in A $= \underbrace{\binom{n}{k}}_{\substack{\text{\# of ways of} \\ \text{choosing } k \\ \text{orders for distributor 1}}} \underbrace{(M-1)^{n-k}}_{\substack{\text{\# of ways of} \\ \text{assigning the} \\ \text{remaining orders} \\ \text{to the} \\ \text{other distribut}}}$

Hence, $P(A) = \dfrac{\binom{n}{k}(M-1)^{n-k}}{M^n}$.

Example 3: A company has 20 new jobs for which it has recruited 20 employees. There are 6 jobs in City 1, 4 jobs in City 2, 5 jobs in City 3 and 5 jobs in City 4. Out of the 20 employees, 4 are friends. Assuming that the company gives no preference to any person in assigning jobs, find the probability that all 4 friends land in the same city.

Solution:

\# of sample points = # of ways of partitioning 20 people into 4 groups of 6, 4, 5, 5 respectively

$$= \frac{20!}{6!\,4!\,5!\,5!}$$

\# of sample points with all 4 friends in City 1

$$= \frac{16!}{2!\,4!\,5!\,5!}$$

\# of sample points with all 4 friends in City 2

$$= \frac{16!}{6!\,5!\,5!}$$

\# of sample points with all 4 friends in City 3

$$= \frac{16!}{6!\,4!\,1!\,5!}$$

\# of sample points with all 4 friends in City 4

$$= \frac{16!}{6!\,4!\,5!\,1!}$$

Hence, P(All friends land in the same city)

$$= \frac{\dfrac{16!}{2!\,4!\,5!\,5!} + \dfrac{16!}{6!\,5!\,5!} + \dfrac{16!}{6!\,4!\,1!\,5!} + \dfrac{16!}{6!\,4!\,5!\,1!}}{\dfrac{20!}{6!\,4!\,5!\,5!}}$$