LECTURE 5

Agenda:

1. Conditional Probability
2. Independence

CONDITIONAL PROBABILITY

Many times, some partial information about the outcome of a random experiment is available and we want to revise our estimates of the chance of an event accordingly.

Definition: Let A and B be two events in a random experiment with sample space S. Then, the probability of the event A given that the event B has occurred is defined as

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

The symbol $P(A|B)$ is read "THE PROBABILITY OF A GIVEN B".

<u>Example:</u> Toss a fair die. Let A denote the event that the outcome is 2, 4 or 6. If somebody asks you to play the following game:

"If event A occurs, you pay me \$10, otherwise I will pay you \$10"

will you play the game?

<u>Ans:</u> $S' = \{1, 2, 3, 4, 5, 6\}$

All outcomes are equally likely, as it is a fair die with no preference for any outcome.

$A = \{2, 4, 6\}$.

Hence, $P(A) = \frac{1}{2}$.

It seems like a fair game and you would bet.

Suppose the die is cast in a secret chamber and you have a helpful source who tells you that the result was 4, 5 or 6. You have a chance to withdraw or remain in the game. What would you do?

Ans: Let $B = \{4, 5, 6\}$. We are told that B has occured. Let us evaluate $P(A \mid B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{4, 6\})}{P(\{4, 5, 6\})} = \frac{2}{3}.$$

Hence, the chance that you will have to pay the other person \$10 is $\frac{2}{3}$ given this additional information. You would withdraw and not play the game.

Conditional probabilities satisfy the 3 axioms of probability.

Result: Let B be an event with $P(B) > 0$. Then,

1. $0 \leq P(A \mid B) \leq 1$ for every event A.
2. $P(S \mid B) = 1$.
3. If $A_1, A_2, A_3, \ldots$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \sum_{i=1}^{\infty} P(A_i \mid B)$$

Proof: (1) Note that $0 \leq P(A \cap B) \leq P(B)$.

Dividing everything by $P(B)$, we get that,

$$0 \leq \frac{P(A \cap B)}{P(B)} \leq 1$$

$$\Rightarrow \quad 0 \leq P(A | B) \leq 1.$$

(2) $$P(S | B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

(3) $$P\left(\bigcup_{i=1}^{\infty} A_i \,\middle|\, B\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

($\because$ By distributive laws)

$$= \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)}$$

($\because$ $A_1 \cap B$, $A_2 \cap B$, $A_3 \cap B$, ---------- are mutually exclusive)

$$= \sum_{i=1}^{\infty} P(A_i | B).$$

INDEPENDENCE

If the extra information provided by knowing that an event B has occured does not change the probability of A, i.e, if $P(A|B) = P(A)$, then the events A and B are said to be independent.

Definition: Two events A and B are said to be independent if

$$P(A \cap B) = P(A)\,P(B).$$

- $P(A \cap B) = P(A)P(B)$ is equivalent to stating that $P(A|B) = P(A)$ or $P(B|A) = P(B)$, if the conditional probabilities $P(A|B)$ or $P(B|A)$ exist.

- Multiplicative rule: If $A_1, A_2, \ldots, A_n$ are n events, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_{n-1}|A_1 \cap A_2 \cap \cdots \cap A_{n-2}) \times P(A_n|A_1 \cap A_2 \cap \cdots A_n)$$

Example: Draw a card from a shuffled 52-card deck with no preference to any card. Let A denote the event that a king is drawn, and let B denote the event that a diamond $\diamond$ is drawn.

$$P(A) = \frac{4}{52}, \qquad P(B) = \frac{13}{52}, \qquad P(A \cap B) = \frac{1}{52}.$$

Hence, $P(A \cap B) = P(A)\,P(B)$. This gives independence of A and B.