LECTURE 6

Agenda:

1. Inclusion - Exclusion principle
2. Theorem of total probability
3. Bayes rule

INCLUSION - EXCLUSION PRINCIPLE

The inclusion - exclusion principle provides an identity for computing the probability of union of a set of events in terms of intersections of various orders of these events

2 events: If A, B are two events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

3 events: If A, B, C are three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

k events: If $A_1, A_2, ---, A_k$ are k events,

$$P(A_1 \cup A_2 \cup --- \cup A_k) = \sum_{i=1}^{k} P(A_i) - \sum_{\substack{\text{All unordered} \\ \text{pairs } (i_1, i_2)}} P(A_{i_1} \cap A_{i_2})$$

$$+ \sum_{\substack{\text{All unordered} \\ \text{triplets } (i_1, i_2, i_3)}} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) -$$

$$--- + (-1)^{k-1} P(A_{i_1} \cap A_{i_2} \cap --- \cap A_{i_k})$$

Example: An absent-minded secretary prepared five letters and envelope to send to five different people. Then he randomly placed letters in the envelopes. A match occurs if the letter and its envelope are addressed to the same person. What is the probability that atleast one of the five letters and envelopes match?

Let, A = Event that atleast one match occurs.

A_i = Event that Letter i is placed in envelope i,
for $i = 1, 2, 3, 4, 5$.

Then, $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$

Note that all arrangements of the 5 letters are equally likely.

Note that the total number of arrangements possible is $5!$
The number of arrangements in the event A_i is $4!$ for $i = 1, 2, 3, 4, 5$ (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2}$ is $3!$ for all unordered pairs (i_1, i_2) (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ is $2!$ for all unordered triplets (i_1, i_2, i_3) (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap$ is $1!$ for all unordered quadruplets (i_1, i_2, i_3, i_4) (Why

The number of arrangements in the event $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A$ is 1.

Hence, by the inclusion-exclusion principle,

$$P(A) = \sum_{i=1}^{5} \frac{4!}{5!} - \sum_{\substack{\text{All unordered pairs} \\ (i_1, i_2)}} \frac{3!}{5!} + \sum_{\substack{\text{All unordered} \\ \text{triplets } (i_1, i_2, i_3)}} \frac{2!}{5!}$$

$$- \sum_{\substack{\text{All unordered quadruplets} \\ (i_1, i_2, i_3, i_4)}} \frac{1!}{5!} + \frac{1}{5!}$$

$$= 5 \cdot \frac{4!}{5!} - \binom{5}{2} \frac{3!}{5!} + \binom{5}{3} \frac{2!}{5!} - \binom{5}{4} \frac{1!}{5!} + \frac{1}{5!}$$

Hence, $P(A) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} = 0.6417$

THEOREM OF TOTAL PROBABILITY

If $B_1, B_2, ---, B_k$ is a collection of mutually exclusive and exhaustive events, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i)\, P(A / B_i)$$

Example: A company buys microchips from three suppliers. Supplier I microchips have 10% chance of being defective, Supplier II microchips have 5% chance and Supplier III microchips have 2% chance of being defective. Suppose 20%, 35% and 45% of the current supply came from Suppliers I, II, III respectively. If a microchip is selected randomly from this supply, what is the probability that it is defective?

B_1 = Event that microchip comes from Supplier I
B_2 = Event that microchip comes from Supplier II
B_3 = Event that microchip comes from Supplier III

A = Event that microchip is defective

It is given that

$$P(A|B_1) = 0.1, \quad P(A|B_2) = 0.05, \quad P(A|B_3) = 0.02$$

Note that B_1, B_2, B_3 are mutually exclusive and exhaustive, since $B_1 \cup B_2 \cup B_3 = S$, and $B_1 \cap B_2 = \phi$, $B_2 \cap B_3 = \phi$ and $B_1 \cap B_3 = \phi$. Also,

$$P(B_1) = 0.2, \quad P(B_2) = 0.35, \quad P(B_3) = 0.45.$$

Hence, by the Theorem of total probability,

$$P(A) = 0.1 \times 0.2 + 0.05 \times 0.35 + 0.02 \times 0.45 = 0.046$$

BAYES RULE

If the events $B_1, B_2, \ldots, B_k$ are mutually exclusive and exhaustive, then for any event A,

$$P(B_i|A) = \frac{P(A|B_i)\,P(B_i)}{\sum_{j=1}^{k} P(A|B_j)\,P(B_j)}.$$

Example: In the previous example, if a randomly selected microchip is defective, what is the probability that it came from supplier II ?

By Bayes rule,

$$P(B_2 \mid A) = \frac{P(A \mid B_2)\, P(B_2)}{\sum_{i=1}^{3} P(A \mid B_i)\, P(B_i)}$$

$$= \frac{0.05 \times 0.35}{0.0465}$$

$$= 0.376 .$$