LECTURE - 7

Agenda:

- The Monty Hall problem
- Other problems

THE MONTY HALL PROBLEM

Consider a game where there are 3 doors. 2 goats and 1 car are placed randomly behind the 3 doors. The player, who cannot see through the doors, is asked to choose a door. The door is kept closed for the time being, and the host (who knows which door has the car) then opens one of the two remaining doors which has a goat (if both remaining doors have goats, he randomly picks one). The player is now given a choice: Open the door which he/she originally chose or switch to the door which still remains closed. He/she wins whatever is behind the door he/she chooses to open. What is the right strategy for the player?

Suppose the player originally picked door 1 and the host opens door 3. Let,

C_i = Event that Car is behind door i, $i = \underline{1}, 2, 3$

H_3 = Event that host opens door 3

We need to evaluate

$$P(C_2 \mid H_3). = P(\text{Car is behind door } 2 \mid \text{Host picks door } 3)$$

Note that $C_1 \cup C_2 \cup C_3 = S$ and C_1, C_2, C_3 are mutually exclusive. Hence, by the law of total probability,

$$P(H_3) = \sum_{i=1}^{3} P(H_3 \mid C_i)\, P(C_i),$$

and by Bayes rule

$$P(C_2 \mid H_3) = \frac{P(H_3 \mid C_2)\, P(C_2)}{\sum_{i=1}^{3} P(H_3 \mid C_i)\, P(C_i)}$$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}}$$

$$= \frac{2}{3}.$$

Hence the right strategy is to always switch doors.

OTHER PROBLEMS

(i) Prove that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

$\binom{n}{r}$ = # of ways of choosing r objects from n objects without replacement, order is not important

Here is another way of counting these ways. Fix one of the objects, call it "J". Every subset of size r chosen either contains "J" or does not contain "J".

$\binom{n-1}{r}$ = # of ways of choosing r objects from n objects WITHOUT choosing "J".

$\binom{n-1}{r-1}$ = # of ways of choosing r objects from n objects WITH "J" being one of them.

(ii) Prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

2^n = # of possible outcomes of n successive tosses.

$\binom{n}{i}$ = # of outcomes with i heads, $i = 0, 1, 2, ---, n-1, n$

Since the number of possible heads is $0, 1, ---, n$, it follows that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

(iii) $$k^r = \sum_{\substack{(r_1, r_2, ---, r_k) \\ \sum_{i=1}^{k} r_i = r,\ r_i \geq 0}} \frac{r!}{r_1! r_2! --- r_k!}$$

k^r = # of ways of choosing r objects from k objects with replacement and order is important

Note that if r_1 is the number of times 1 is chosen, r_2 is the number of times 2 is chosen and so on, then $r_1 + r_2 + --- + r_k = r$, $r_i \geq 0$. However, there are $\frac{r!}{r_1! r_2! --- r_k!}$ such arrangements for any given k-type $(r_1, r_2, ---, r_k)$ of counts.

It follows that

$$k^r = \sum_{\substack{(r_1, r_2, ---, r_k) \\ \sum_{i=1}^{k} r_i = r}} \frac{r!}{r_1! r_2! --- r_k!}$$

(iv) For $N \geq n$, $\binom{N}{n} = \sum_{i=0}^{n} \binom{M}{i} \binom{N-M}{n-i}$

where $0 < M < N$.

Suppose you have N objects and we want to choose n objects without replacement (order is not important) from them. The # of ways is $\binom{N}{n}$.

Divide the N objects into two groups of size M and $N-M$. The number of ways of choosing n objects from N objects such that i come from the first group and $n-i$ come from the second group is $\binom{M}{i}\binom{N-M}{n-i}$. Since the values that i can take vary from 0 to n, it follows that

$$\binom{N}{n} = \sum_{i=0}^{n} \binom{M}{i}\binom{N-M}{n-i}.$$