LECTURE - 9

Agenda:

1. Properties of distribution function
2. Expected values
3. Variance / standard deviation

PROPERTIES OF DISTRIBUTION FUNCTION

Let us recollect that if X is a random variable, then its distribution function $F_X : \mathbb{R} \to [0, 1]$ is defined by

$$F_X(b) = P(X \le b) \quad \text{for all } b \in \mathbb{R}.$$

$\boxed{(i)}$ $\lim_{b \to -\infty} F_X(b) = 0$

Proof:

$$\lim_{b \to -\infty} F_X(b) = \lim_{b \to -\infty} P(X \le b)$$
$$= P(X \le -\infty)$$
$$= 0.$$

$\boxed{(ii)}$ $\lim_{b \to \infty} F_X(b) = \lim_{b \to \infty} P(X \le b)$

$$= P(X \le \infty)$$
$$= 1.$$

$\boxed{\text{(iii)}}$ F_X is a non-decreasing function.

Proof: Let $a < b$. Then,

$$F_X(a) = P(X \leq a) \quad , \quad F_X(b) = P(X \leq b).$$

Since the event $\{X \leq a\}$ is contained in the event $\{X \leq b\}$,

$$P(X \leq a) \leq P(X \leq b).$$

Hence $F_X(a) \leq F_X(b)$.

$\boxed{\text{(iv)}}$ F_X is right-hand continuous, i.e.,

$$\lim_{h \to 0^+} F_X(b+h) = F_X(b).$$

Proof: Note that,

$$\{X \leq b\} = \bigcap_{m \geq 1} \left\{X \leq b + \frac{1}{m}\right\}$$

Hence, $$\lim_{m \to \infty} P\left(X \leq b + \frac{1}{n}\right) = \lim_{n \to \infty} P\left(\bigcap_{m \geq 1}^{n} \left\{X \leq b + \frac{1}{m}\right\}\right)$$

$$= P\left(\bigcap_{m \geq 1} \left\{X \leq b + \frac{1}{m}\right\}\right)$$

$$= P(X \leq b)$$

A similar argument holds for any sequence

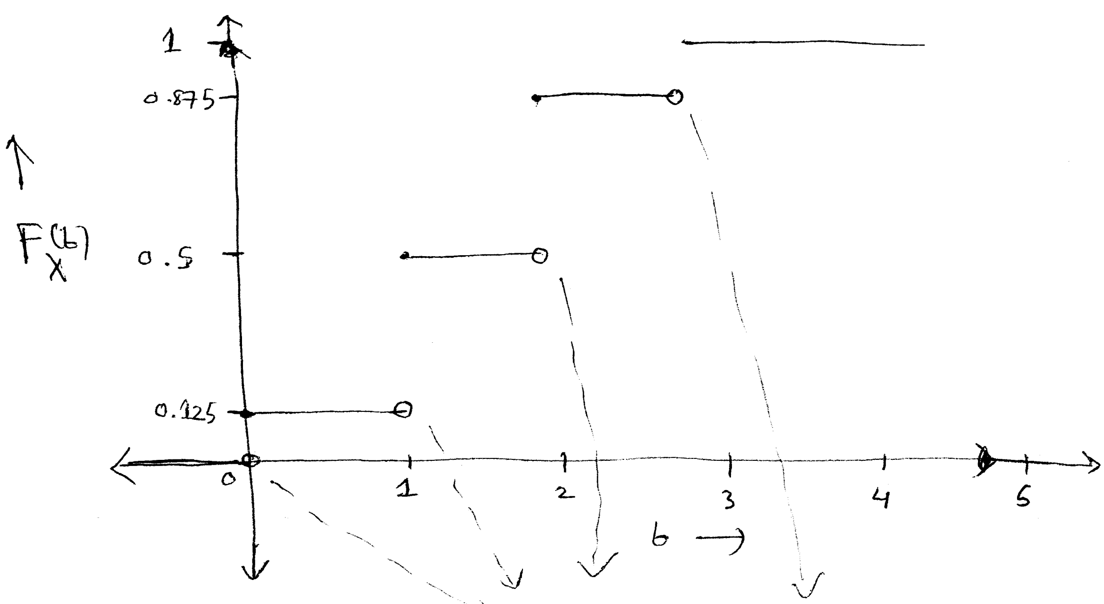
However, it is not true that F_X is left-hand continuous, especially for discrete random variables.

<u>Example:</u> Let X be the number of heads for 3 tosses of a fair coin. Then $P(X=0) = \frac{1}{8}$,

$P(X=1) = \frac{3}{8}$, $P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$.

Then, $F_X(b) = \sum_{x \in \{0,1,2,3\} : x \le b} P(X = x)$. Hence,

$$F_X(b) = \begin{cases} 0 & \text{if } b < 0, \\ 0.125 & \text{if } 0 \le b < 1, \\ 0.5 & \text{if } 1 \le b < 2, \\ 0.875 & \text{if } 2 \le b < 3, \\ 1 & \text{if } 3 \le b. \end{cases}$$



FACT: Any function satisfying properties (i), (ii), (iii) and (iv) is a distribution function of some random variable.

EXPECTED VALUES

Suppose we are interested in a random variable X arising out of a random experiment. Based on our understanding of the random experiment, we have a probability model. Often, we want to summarize our understanding of the random variable in one number; "The expected value" of the random variable.

DEFINITION: The "expected value" of a discrete random variable X with probability mass function p_X is given by

$$E(X) = \sum_{x \in \mathcal{X}} x \, p_X(x) = \sum_{x \in \mathcal{X}} x \, P(X = x).$$

It is also understood as our estimate of the "average" value that the random variable will take.

NOTE: The expected value of a discrete random variable is defined only if $\sum_{x \in \mathcal{X}} |x| \, P(X=x) < \infty$.

Example: Consider the following game. We toss a six-faced die 2 times. If the sum of the two values is 3 or lower, we have to pay 10 dollars. If the sum of the two values is 4, 5 or 6 we pay 4 dollars. If the sum of the two values is 7, 8 or 9 we earn 4 dollars. If the sum of the two values is 10, 11 or 12 we earn 10 dollars. What are the expected winnings?

Let X = Sum of two values on the die

$$P(X = 2 \text{ or } 3) = \frac{3}{36} = \frac{1}{12}$$

$$P(X = 4, 5 \text{ or } 6) = \frac{12}{36} = \frac{1}{3}$$

$$P(X = 7, 8 \text{ or } 9) = \frac{15}{36} = \frac{5}{12}$$

$$P(X = 10, 11 \text{ or } 12) = \frac{6}{36} = \frac{1}{6}$$

Let W = Winnings in the game

$P(W = -10) = \frac{1}{12}$, $P(W = -4) = \frac{1}{3}$, $P(W = 4) = \frac{5}{12}$,

$P(W = 10) = \frac{1}{6}$.

$$E[W] = \left(-10 \times \frac{1}{12}\right) + \left(-4 \times \frac{1}{3}\right) + \left(4 \times \frac{5}{12}\right) + \left(10 \times \frac{1}{6}\right)$$

$$= \frac{-10 - 16 + 20 + 20}{12}$$

$$= \frac{7}{6}$$

RESULT: If X is a discrete random variable with probability distribution p_X and if $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function, then

$$E(g(X)) = \sum_{x \in \mathcal{X}} g(x) \, p_X(x).$$

If $g(x) = x^2$, then $E[g(W)]$ is given by

$$E[W^2] = 100 \times \frac{1}{12} + 16 \times \frac{1}{3} + 16 \times \frac{5}{12} + 100 \times \frac{1}{6}$$

$$= 25 + 12$$

$$= 37.$$

VARIANCE

DEFINITION: The variance of a random variable X with expected value μ is given by

$$V(X) = E(X-\mu)^2 = \sum_{x \in \mathcal{X}} (x-\mu)^2 p_X(x).$$