Homework #1.

$\boxed{8.8/p.394}$

(a) $E(\hat\theta_1) = E(Y_1) = \theta$

$E(\hat\theta_2) = E\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{2}E(Y_1) + \frac{1}{2}E(Y_2) = \frac{1}{2}\theta + \frac{1}{2}\theta = \theta$

$E(\hat\theta_3) = E\left(\frac{Y_1+2Y_2}{3}\right) = \frac{1}{3}E(Y_1) + \frac{2}{3}E(Y_2) = \frac{1}{3}\theta + \frac{2}{3}\theta = \theta$

$E(\hat\theta_5) = E(\bar Y) = E\left(\frac{Y_1+Y_2+Y_3}{3}\right) = \frac{1}{3}E(Y_1) + \frac{1}{3}E(Y_2) + \frac{1}{3}E(Y_3) =$

$= \frac{1}{3}\theta + \frac{1}{3}\theta + \frac{1}{3}\theta = \theta$

$\hat\theta_4 = Y_{(1)} = \min(Y_1, Y_2, Y_3)$

$F(y) = 1 - e^{-y/\theta},\ y>0$

$g_{(1)}(y) = n\,F(y)^{n-1} f(y) = n\,e^{-y(n-1)/\theta} \cdot \frac{1}{\theta} \cdot e^{-y/\theta} = \begin{cases} \frac{n}{\theta} e^{-ny/\theta}, & y>0 \\ 0, & \text{otherwise} \end{cases}$

$\hat\theta_4 = Y_{(1)} \sim$ exponential $(\theta/3)$ since $n=3$

$E(\hat\theta_4) = \frac{\theta}{3}$

$\hat\theta_1, \hat\theta_2, \hat\theta_3, \hat\theta_5$ are unbiased ; $\hat\theta_4$ is biased

(b) $V(\hat\theta_1) = V(Y_1) = \theta^2$

$V(\hat\theta_2) = V\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{2^2}\left(V(Y_1)+V(Y_2)\right) = \frac{1}{4}(\theta^2+\theta^2) = \frac{\theta^2}{2}$

$V(\hat\theta_3) = V\left(\frac{Y_1+2Y_2}{3}\right) = \frac{1}{3^2}\left(V(Y_1)+2^2V(Y_2)\right) = \frac{1}{9}(\theta^2+4\theta^2) = \frac{5\theta^2}{9}$

$V(\hat\theta_5) = V\left(\frac{\sum_{i=1}^{3} Y_i}{3}\right) = \frac{\sum_{i=1}^{3} V(Y_i)}{3^2} = \frac{3\theta^2}{9} = \frac{\theta^2}{3}$

$\hat\theta_5 = \bar Y$ has the smallest variance

$\boxed{8.9/p.395}$

$Y \sim$ exponential $(\theta+1) \Rightarrow E(Y) = \theta+1$

$E(\bar{Y}) = \theta + 1$ (i.e. $\bar{Y}$ is an unbiased estimator for the mean).

$\Rightarrow$ $\bar{Y} - 1$ is an unbiased estimator for θ, since

$E(\bar{Y}-1) = E(\bar{Y}) - 1 = \theta + 1 - 1 = \theta$

$\boxed{8.10/p.395}$

ⓐ $E(\bar{Y}) = E\left(\frac{\sum_{i=1}^{n} Y_i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\cdot n\cdot\lambda = \lambda$

$\Rightarrow$ $\hat{\lambda} = \bar{Y}$ is an unbiased estimator for λ

ⓑ $E(C) = E(3Y + Y^2) = 3E(Y) + E(Y^2) = 3E(Y) + [E(Y)]^2 + V(Y)$

$= 3\lambda + \lambda^2 + \lambda = 4\lambda + \lambda^2$

ⓒ $E(\bar{Y}) = \lambda$, $E(\bar{Y})^2 = V(\bar{Y}) + (E(\bar{Y}))^2 = \frac{V(Y)}{n} + (E(\bar{Y}))^2 = \frac{\lambda}{n} + \lambda^2$

consider $\hat{\theta} = \bar{Y}^2 + \bar{Y}(4 - \frac{1}{n})$

$E(\hat{\theta}) = E(\bar{Y}^2) + (4 - \frac{1}{n})E(\bar{Y}) = \frac{\lambda}{n} + \lambda^2 + (4 - \frac{1}{n})\lambda = 4\lambda + \lambda^2$

$\Rightarrow$ $\hat{\theta} = \bar{Y}^2 + \bar{Y}(4 - \frac{1}{n})$ is an unbiased estimator of $E(C)$

$\boxed{8.11/p.395}$

third central moment $= E(Y-\mu)^3 = E(Y-3)^3$ since $\mu = 3$

$E(\hat{\theta}_2) = E(Y^2)$ and $E(\hat{\theta}_3) = E(Y^3)$

$E(Y-3)^3 = E(Y^3 - 3Y^2\cdot 3 + 3Y\cdot 3^3 - 3^3) = E(Y^3) - 9E(Y^2) + 3^3E(Y) - 3^3$

$= E(Y^3) - 9E(Y^2) + 3\cdot 3^3 - 3^3 = E(Y^3) - 9E(Y^2) + 54$

$= E(Y^3) - 9E(Y^2) + 3\cdot3^3 - 3^3$ …

Since $E(Y^3) = E(\hat{\theta}_3)$ and $E(Y^2) = E(\hat{\theta}_2)$

$\Rightarrow$ $\hat{\theta}_3 - 9\hat{\theta}_2 + 54$ is an unbiased estimator for $E(Y-3)^3$ since $E(\hat{\theta}_3 - 9\hat{\theta}_2 + 54) = E(Y-3)^3$

8.12/p.395

$$f(y) = \frac{1}{\theta+1-\theta} = \begin{cases} 1 & , \text{ if } y \in (\theta, \theta+1) \\ 0 & , \text{ otherwise} \end{cases}$$

(a) $E(Y_i) = \frac{\theta + \theta + 1}{2} = \theta + \frac{1}{2}$

$E(\bar{Y}) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \cdot E(Y_i) = E(Y_i) = \theta + \frac{1}{2}$

$B(\bar{Y}) = E(\bar{Y}) - \theta = \theta + \frac{1}{2} - \theta = \frac{1}{2}$

(b) Since $\text{Bias}(\bar{Y}) = \frac{1}{2} \Rightarrow \bar{Y} - \frac{1}{2}$ is unbiased (i.e $E(\bar{Y} - \frac{1}{2}) = \theta$)

(c) $MSE(\bar{Y}) = V(\bar{Y}) + [B(\bar{Y})]^2$

$V(\bar{Y}) = \frac{V(Y_i)}{n} = \frac{(\theta + 1 - \theta)^2}{12} \cdot \frac{1}{n} = \frac{1}{12n}$

$\Rightarrow MSE(\bar{Y}) = \frac{1}{12n} + \left(\frac{1}{2}\right)^2 = \frac{1}{12n} + \frac{1}{4}$

8.14 / p.395

$E(Y) = \int_0^\theta \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \cdot y \, dy = \int_0^\theta \frac{\alpha y^\alpha}{\theta^\alpha} dy = \frac{\alpha y^{\alpha+1}}{(\alpha+1)\theta^\alpha} \Big|_0^\theta = \frac{\alpha \theta^{\alpha+1}}{(\alpha+1)\theta^\alpha} = \frac{\alpha\theta}{\alpha+1}$

$E(Y^2) = \int_0^\theta \frac{\alpha y^{\alpha-1}}{\theta^\alpha} y^2 \, dy = \int_0^\theta \frac{\alpha y^{\alpha+1}}{\theta^\alpha} dy = \frac{\alpha y^{\alpha+2}}{(\alpha+2)\theta^\alpha} \Big|_0^\theta = \frac{\alpha\theta^{\alpha+2}}{(\alpha+2)\theta^\alpha} = \frac{\alpha\theta^2}{\alpha+2}$

$F(y) = \int_0^y \frac{\alpha t^{\alpha-1}}{\theta^\alpha} dt = \frac{\alpha t^\alpha}{\alpha\theta^\alpha} \Big|_0^y = \frac{y^\alpha}{\theta^\alpha} = \left(\frac{y}{\theta}\right)^\alpha$

$$\Rightarrow F(y) = \begin{cases} 0 & , y < 0 \\ \left(\frac{y}{\theta}\right)^\alpha & , 0 \le y \le \theta \\ 1 & , y > \theta \end{cases}$$

$\hat{\theta} = Y_{(n)} = \max(Y_1, Y_2, \dots Y_n)$

$\Rightarrow f_{Y_{(n)}} = n [F(y)]^{n-1} f(y) = n \cdot \left(\frac{y}{\theta}\right)^{\alpha \cdot (n-1)} \cdot \frac{\alpha y^{\alpha-1}}{\theta^\alpha} =$

$$= \begin{cases} \frac{n\alpha y^{n\alpha-1}}{\theta^{n\alpha}} & , 0 \le y \le \theta \\ 0 & , \text{otherwise} \end{cases}$$

$\Rightarrow Y_{(n)} = \hat{\theta}$ a power distribution with $\alpha^* = n\alpha$ and $\theta^* = \theta$

(a) $E(\hat{\Theta}) = \left(\frac{\alpha^*}{\alpha^*+1}\right)\Theta^* = \frac{n\alpha}{n\alpha+1}\cdot\Theta \neq \Theta$ (i.e biased)

(b) Consider $\frac{n\alpha+1}{n\alpha}\hat{\Theta}$ since $E\left(\frac{n\alpha+1}{n\alpha}\hat{\Theta}\right) = \frac{n\alpha+1}{n\alpha}E(\hat{\Theta}) =$

$$= \frac{n\alpha+1}{n\alpha}\cdot\frac{n\alpha}{n\alpha+1}\cdot\Theta = \Theta$$

(c) $MSE(\hat{\Theta}) = E(\hat{\Theta}-\Theta)^2 = E(\hat{\Theta}^2 - 2\hat{\Theta}\Theta + \Theta^2) = E(\hat{\Theta}^2) - 2\Theta E(\hat{\Theta}) + \Theta^2$

$$= \frac{\alpha^*}{\alpha^*+2}\cdot(\Theta^*)^2 - 2\Theta\frac{\alpha^*}{\alpha^*+1}\Theta^* + \Theta^2 = \frac{n\alpha}{n\alpha+2}\cdot\Theta^2 - 2\frac{n\alpha}{n\alpha+1}\cdot\Theta^2 + \Theta^2 =$$

$$= \frac{\left[(n\alpha+1)n\alpha - 2(n\alpha+2)n\alpha + (n\alpha+1)(n\alpha+2)\right]\Theta^2}{(n\alpha+2)(n\alpha+1)} =$$

$$= \frac{(n^2\alpha^2 + n\alpha - 2n^2\alpha^2 - 4n\alpha + n^2\alpha^2 + 2n\alpha + n\alpha + 2)\Theta^2}{(n\alpha+2)(n\alpha+1)} = \frac{2\Theta^2}{(n\alpha+2)(n\alpha+1)}$$

$\boxed{8.15/p.395}$

$$F(y) = \int_\beta^y 3\beta^3 t^{-4}\,dt = \frac{3\beta^3 t^{-3}}{-3}\Bigg|_\beta^y = -\beta^3 y^{-3} + \beta^3\beta^{-3} = 1 - \left(\frac{\beta}{y}\right)^3$$

$$\Rightarrow F(y) = \begin{cases} 0, & y < \beta \\ 1 - \left(\frac{\beta}{y}\right)^3, & y \geq \beta \end{cases}$$

$Y_{(1)} = \hat{\beta} = \min(Y_1, Y_2, \dots, Y_n)$

$$f(y_{(1)}) = n\left[1 - F(y)\right]^{n-1} f(y) = n\left(\frac{\beta}{y}\right)^{3(n-1)} 3\beta^3 y^{-4} = \begin{cases} 3n\beta^{3n} y^{-3n-1}, & y \geq \beta \\ 0, & \text{otherwise} \end{cases}$$

$$E(\hat{\beta}) = \int_\beta^\infty 3n\beta^{3n} y^{-3n-1}\cdot y\,dy = 3n\beta^{3n}\int_\beta^\infty y^{-3n}\,dy = 3n\beta^{3n}\frac{y^{-3n+1}}{-3n+1}\Bigg|_\beta^\infty$$

$$= 3n\beta^{3n}\frac{\beta^{-3n+1}}{3n-1} = \frac{3n}{3n-1}\cdot\beta$$

$$E(\hat{\beta}^2) = \int_\beta^\infty 3n\beta^{3n} y^{-3n-1} y^2\,dy = 3n\beta^{3n}\int_\beta^\infty y^{-3n+1}\,dy =$$

$$E(\hat{\beta}^2) = 3n\,\beta^{3n} \left. \frac{y^{-3n+2}}{-3n+2} \right|_{\beta}^{\infty} = 3n\,\beta^{3n} \frac{\beta^{-3n+2}}{3n-2} = \frac{3n}{3n-2}\beta^2$$

a) $$B(\hat{\beta}) = E(\hat{\beta}) - \beta = \frac{3n}{3n-1}\beta - \beta = \frac{3n\beta - 3n\beta + \beta}{3n-1} = \frac{\beta}{3n-1}$$

b) $$MSE(\hat{\beta}) = E(\hat{\beta}-\beta)^2 = E(\hat{\beta}^2 - 2\hat{\beta}\beta + \beta^2) = E(\hat{\beta}^2) - 2\beta E(\hat{\beta}) + \beta^2 =$$

$$= \frac{3n}{3n-2}\beta^2 - 2\beta^2\frac{3n}{3n-1} + \beta^2 = \frac{[3n(3n-1) - 2\cdot 3n(3n-2) + (3n-2)(3n-1)]\beta^2}{(3n-2)(3n-1)}$$

$$= \frac{[9n^2 - 3n - 18n^2 + 12n + 9n^2 - 3n - 6n + 2]\beta^2}{(3n-2)(3n-1)} = \frac{2\beta^2}{(3n-2)(3n-1)}$$