HW #3

$\boxed{8.39/p.409}$

$Y \sim \text{gamma}(\alpha=2, \beta)$

$\frac{2Y}{\beta} \sim \chi^2(4\,df)$

① $\frac{2Y}{\beta}$ is our pivotal quantity (its distribution is independent of β)

② Need to find a and b such that:

$P(a \le \frac{2Y}{\beta} \le b) = 0.9 \quad \Rightarrow \alpha = 0.1$

$\frac{2Y}{\beta} \sim \chi^2(4\,df) \Rightarrow$ use Table 6 on pages 880-881

$\chi^2_{0.1/2} = \chi^2_{0.05} = 9.48773 = b$

$\chi^2_{(1-0.1/2)} = \chi^2_{0.95} = 0.71072 = a$

$P(0.71072 \le \frac{2Y}{\beta} \le 9.48773) = 0.9$

③

$0.71072 \le \frac{2Y}{\beta} \le 9.48773 \Leftrightarrow$

$\Leftrightarrow \frac{0.71072}{2Y} \le \frac{1}{\beta} \le \frac{9.48773}{2Y} \Leftrightarrow$

$\Leftrightarrow \frac{2Y}{9.48773} \le \beta \le \frac{2Y}{0.71072}$

$\Rightarrow$ a 90% CI for β is $\left[\frac{2Y}{9.48773}, \frac{2Y}{0.71072}\right]$

$\boxed{8.41/p.409-410}$

$Y \sim \text{normal}(0, \sigma^2)$,

$\frac{Y^2}{\sigma^2} \sim \chi^2(1\,df)$

$\boxed{a}$ ① $\frac{Y^2}{\sigma^2}$ is our pivotal quantity (independent of σ^2)

② Find a and b such that

$P(a \le \frac{Y^2}{\sigma^2} \le b) = 0.95 \Rightarrow \alpha = 0.05$

$\frac{Y^2}{\sigma^2} \sim \chi^2(1df) \Rightarrow$ use Table 6 on pages 850-851

$\chi^2_{0.05/2} = \chi^2_{0.025} = 5.02389 = b$

$\chi^2_{(1-0.05/2)} = \chi^2_{0.975} = 0.00098 = a$

$P(0.00098 \le \frac{Y^2}{\sigma^2} \le 5.02389) = 0.95$

3 $0.00098 \le \frac{Y^2}{\sigma^2} \le 5.02389 \Leftrightarrow \frac{0.00098}{Y^2} \le \frac{1}{\sigma^2} \le \frac{5.02389}{Y^2} \Leftrightarrow$

$\Leftrightarrow \frac{Y^2}{5.02389} \le \sigma^2 \le \frac{Y^2}{0.00098}$

$\Rightarrow$ a 95% CI for σ^2 is $\left[\frac{Y^2}{5.02389}, \frac{Y^2}{0.00098}\right]$

$\boxed{b}$ 95% upper confidence limit for σ^2

1 $\frac{Y^2}{\sigma^2}$ - pivotal quantity

2 find b such that

$P(b \le \frac{Y^2}{\sigma^2}) = 0.95 \Rightarrow \alpha = 0.05$

$\chi^2_{(1-0.05)} = \chi^2_{0.95} = 0.00393$ (Table 6 p. 850, df=1)

$P(0.00393 \le \frac{Y^2}{\sigma^2}) = 0.95$

3 $0.00393 \le \frac{Y^2}{\sigma^2} \Leftrightarrow \frac{0.00393}{Y^2} \le \frac{1}{\sigma^2} \Leftrightarrow \sigma^2 \le \frac{Y^2}{0.00393}$

$\Rightarrow$ the 95% upper confidence limit for σ^2 is $\frac{Y^2}{0.00393}$

$\boxed{c}$ 95% lower confidence limit for σ^2

1 $\frac{Y^2}{\sigma^2}$ - pivotal quantity

2 find a such that

$P(\frac{Y^2}{\sigma^2} \le a) = 0.95 \Rightarrow \alpha = 0.05$

Table 6/p. 850 (df=1) $\Rightarrow \chi^2_{0.05} = 3.84146$

$P(\frac{Y^2}{\sigma^2} \le 3.84146) = 0.95$

3 $\frac{Y^2}{\sigma^2} \le 3.84146 \iff \frac{1}{\sigma^2} \le \frac{3.84146}{Y^2} \iff \frac{Y^2}{3.84146} \le \sigma^2$

$\Rightarrow$ The 95% lower confidence limit for σ^2 is $\frac{Y^2}{3.84146}$

$\boxed{8.42/p.410}$

Taking square-roots of the boundaries for $\sigma^2 \Rightarrow$ boundaries for σ

$\boxed{a}$ a 95% CI for σ is $\left[\sqrt{\frac{Y^2}{5.02389}}, \sqrt{\frac{Y^2}{0.00098}}\right] \Rightarrow \left[\frac{Y}{2.2414}, \frac{Y}{0.0313}\right]$

$\boxed{b}$ the 95% upper confidence limit for σ is

$$\sqrt{\frac{Y^2}{0.00393}} = \frac{Y}{0.06269}$$

$\boxed{c}$ the 95% lower confidence limit for σ is

$$\sqrt{\frac{Y^2}{3.84146}} = \frac{Y}{1.96}$$

$\boxed{8.47/p.410-411}$

$\boxed{b}$

1 by $\boxed{a}$ $2\sum_{i=1}^{n} Y_i/\theta$ is a pivotal quantity and has a χ^2 distribution with $2n$ df (and independent of θ)

2 need a and b such that

$P\left(a \le 2\sum_{i=1}^{n} Y_i/\theta \le b\right) = 0.95 \Rightarrow \alpha = 0.05$

$\chi^2_{0.05/2} = \chi^2_{0.025} = b$

$\chi^2_{(1-0.05/2)} = \chi^2_{0.975} = a$

$P\left(\chi^2_{0.975} \le 2\sum_{i=1}^{n} Y_i/\theta \le \chi^2_{0.025}\right) = 0.95$

3 $$\chi^2_{0.975} \le \frac{2\sum_{i=1}^{n} Y_i}{\theta} \le \chi^2_{0.025} \iff \frac{\chi^2_{0.975}}{2\sum_{i=1}^{n} Y_i} \le \frac{1}{\theta} \le \frac{\chi^2_{0.025}}{2\sum_{i=1}^{n} Y_i} \iff$$

$$\iff \frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.025}} \le \theta \le \frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.975}}$$

$\Rightarrow$ a 95% CI for θ is $\left[\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.025}} , \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.975}} \right]$

$\boxed{c}$ $n=7$, $\bar{Y} = 4.77$

$\Rightarrow 2\sum_{i=1}^{7} Y_i/\theta \sim \chi^2(2\cdot 7 = 14 \text{ df})$

also $\sum_{i=1}^{n} Y_i = n\bar{Y} \Rightarrow \sum_{i=1}^{7} Y_i = 7 \times 4.77$

from Table 6/p.850 (df = 14) $\Rightarrow \chi^2_{0.025} = 26.119$ and $\chi^2_{0.975} = 5.62872$

$\Rightarrow$ a 95% CI for θ is $\left[\dfrac{2\times 7\times 4.77}{26.119} , \dfrac{2\times 7\times 4.77}{5.62872} \right]$ or $[2.55676, 11.86415]$

$\boxed{8.48/p.411}$

$\boxed{b}$

(1) by $\boxed{a}$ $2\sum_{i=1}^{n} Y_i/\beta$ is a pivotal quantity and has a χ^2 distribution with $4n$ df

(2) need a and b such that

$P(a \le 2\sum_{i=1}^{n} Y_i/\beta \le b) = 0.95 \quad \Rightarrow \alpha = 0.05$

$\chi^2_{0.05/2} = \chi^2_{0.025} = b$

$\chi^2_{(1-0.05/2)} = \chi^2_{0.975} = a$

$P(\chi^2_{0.975} \le 2\sum_{i=1}^{n} Y_i/\beta \le \chi^2_{0.025}) = 0.95$

(3) $\chi^2_{0.975} \le 2\dfrac{\sum_{i=1}^{n} Y_i}{\beta} \le \chi^2_{0.025} \iff \dfrac{\chi^2_{0.975}}{2\sum_{i=1}^{n} Y_i} \le \dfrac{1}{\beta} \le \dfrac{\chi^2_{0.025}}{2\sum_{i=1}^{n} Y_i} \iff$

$\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.025}} \le \beta \le \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.975}} \Rightarrow$ a 95% CI for β is $\left[\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.025}} , \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{0.975}} \right]$

$\boxed{c}$ $n=5 \Rightarrow 4\times n = 4\times 5 = 20$ df. (use Table 6/p.850)

$\Rightarrow \chi^2_{0.025} = 34.1696$, $\chi^2_{0.975} = 9.59083$

$\bar{Y} = 5.39 \Rightarrow \sum_{i=1}^{5} Y_i = 5\times\bar{Y} = 5\times 5.39$

$\Rightarrow$ a 95% CI for β is $\left[\dfrac{2\times 5\times 5.39}{34.1696} , \dfrac{2\times 5\times 5.39}{9.59082} \right]$ or $[1.57743, 5.61995]$