hw4 solutions

8.56

a. Since $n = 800$ is large enough, by CLT the distribution of $\frac{\hat{p}-p}{\sigma_{\hat{p}}}$ is approximately $N(0,1)$, thus

$$P(-z_{1-\alpha/2} \le \frac{\hat{p}-p}{\sigma_{\hat{p}}} \le z_{1-\alpha/2}) \approx 1-\alpha.$$

$$-z_{1-\alpha/2} \le \frac{\hat{p}-p}{\sigma_{\hat{p}}} \le z_{1-\alpha/2} \Leftrightarrow \hat{p} - z_{1-\alpha/2}\sigma_{\hat{p}} \le p \le \hat{p} + z_{1-\alpha/2}\sigma_{\hat{p}},$$

hence, a $(1-\alpha)100\%$ CI for p is $[\hat{p} - z_{1-\alpha/2}\sigma_{\hat{p}}, \hat{p} + z_{1-\alpha/2}\sigma_{\hat{p}}]$.
$\alpha = .02 \Rightarrow z_{1-\alpha/2} = 2.326$, $\hat{p} = .45$, $\sigma_{\hat{p}} = \sqrt{\hat{p}(1-\hat{p})/n} = \sqrt{.45(1-.45)/800} = 0.01759$, thus a 98% CI for p is $[.45 - 2.326 \times 0.01759, .45 + 2.326 \times 0.01759]$, or $[.40909, .49091]$.

b. The value .5 is not contained in the above interval. Thus, there is no evidence that a majority of adults feel that movies are getting better.

8.59

a. Same as above, we get a $(1-\alpha)100\%$ CI for p is $[\hat{p} - z_{1-\alpha/2}\sigma_{\hat{p}}, \hat{p} + z_{1-\alpha/2}\sigma_{\hat{p}}]$.
$\alpha = .01 \Rightarrow z_{1-\alpha/2} = 1.645$, $\hat{p} = .78$, $\sigma_{\hat{p}} = \sqrt{\hat{p}(1-\hat{p})/n} = \sqrt{.78(1-.78)/1030} = 0.01291$, thus a 90% CI for p is $[.78 - 1.645 \times 0.01291, .78 + 1.645 \times 0.01291]$, or $[.75876, .80124]$.

b. Since $.75 < .75876$, there is evidence that the true population is greater than 75%.

8.85

See the lecture notes, we get a $(1-\alpha)100\%$ CI for $\mu_1 - \mu_2$ is

$$[(\bar{x}-\bar{y}) - t_{n_1+n_2-2,1-\alpha/2}S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}, (\bar{x}-\bar{y}) + t_{n_1+n_2-2,1-\alpha/2}S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}].$$

$n_1 = 16, \bar{x} = 11, S_1 = 6, n_2 = 20, \bar{y} = 12, S_2 = 8 \Rightarrow$

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{(16-1)6^2 + (20-1)8^2}{16+20-2} = 51.64706.$$

$\alpha = .05 \Rightarrow t_{n_1+n_2-2,1-\alpha/2} \approx z_{1-\alpha/2} = 1.96$, thus a 95% CI for $\mu_1 - \mu_2$ is

$$[(11-12) - 1.96\sqrt{51.64706(\frac{1}{16}+\frac{1}{20})}, (11-12) + 1.96\sqrt{51.64706(\frac{1}{16}+\frac{1}{20})}],$$

or $[-5.72449, 3.72449]$.

8.92

We are using the same method as above, $n_1 = 4, \bar{x} = .22, S_1^2 = .001, n_2 = 5, \bar{y} = .17, S_2^2 = .002 \Rightarrow$

$$S_p^2 = \frac{(4-1).001 + (5-1).002}{4+5-2} = \frac{.011}{7}.$$

$\alpha = .05 \Rightarrow t_{n_1+n_2-2,1-\alpha/2} = t_{7,0.975} = 2.365$, thus a 95% CI for $\mu_1 - \mu_2$ is

$$[(.22-.17) - 2.365\sqrt{\frac{.011}{7}(\frac{1}{4}+\frac{1}{5})}, (.22-.17) + 2.365\sqrt{\frac{.011}{7}(\frac{1}{4}+\frac{1}{5})}],$$

or $[-.01289, .11289]$.

9.1

See Exercise 8.8.

$V(\hat{\theta}_1) = \theta^2$, $V(\hat{\theta}_2) = \frac{\theta^2}{2}$, $V(\hat{\theta}_3) = \frac{5\theta^2}{9}$, $V(\hat{\theta}_5) = \frac{\theta^2}{3}$. Since all are unbiased estimators of θ, $MSE(\hat{\theta}_1) = \theta^2$, $MSE(\hat{\theta}_2) = \frac{\theta^2}{2}$, $MSE(\hat{\theta}_3) = \frac{5\theta^2}{9}$, $MSE(\hat{\theta}_5) = \frac{\theta^2}{3}$.

$$eff(\hat{\theta}_1, \hat{\theta}_5) = \frac{MSE(\hat{\theta}_5)}{MSE(\hat{\theta}_1)} = \frac{\frac{\theta^2}{3}}{\theta^2} = \frac{1}{3}$$

$$eff(\hat{\theta}_2, \hat{\theta}_5) = \frac{MSE(\hat{\theta}_5)}{MSE(\hat{\theta}_2)} = \frac{\frac{\theta^2}{3}}{\frac{\theta^2}{2}} = \frac{2}{3}$$

$$eff(\hat{\theta}_3, \hat{\theta}_5) = \frac{MSE(\hat{\theta}_5)}{MSE(\hat{\theta}_3)} = \frac{\frac{\theta^2}{3}}{\frac{5\theta^2}{9}} = \frac{3}{5}$$

9.4

See book pages 333-334.

$$f(y_{(1)}) = n[1 - F(y)]^{n-1} f(y),$$
$$f(y_{(n)}) = n[F(y)]^{n-1} f(y).$$

Since $Y_1 \sim uniform(0, \theta)$,

$$f(y) = \begin{cases} \frac{1}{\theta}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}$$

$$F(y) = \begin{cases} 0, & y \le 0 \\ \frac{y}{\theta}, & 0 < y < \theta \\ 1, & y \ge \theta \end{cases}$$

$$f(y_{(1)}) = \begin{cases} n[1 - \frac{y}{\theta}]^{n-1}\frac{1}{\theta} = \frac{n(\theta-y)^{n-1}}{\theta^n}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
E(y_{(1)}) &= \int_0^\theta f(y_{(1)}) y \, dy \\
&= \int_0^\theta \frac{n(\theta - y)^{n-1}}{\theta^n} y \, dy \\
&= \int_0^\theta \frac{1}{\theta^n} [-(\theta - y)^n]' y \, dy \\
&= -\frac{y}{\theta^n} (\theta - y)^n |_{y=0}^\theta + \int_0^\theta \frac{1}{\theta^n} (\theta - y)^n \, dy \\
&= 0 - \frac{1}{\theta^n} \frac{(\theta - y)^{n+1}}{n+1} \Big|_{y=0}^\theta \\
&= \frac{\theta}{n+1}
\end{aligned}$$

$$\begin{aligned}
E(y_{(1)}^2) &= \int_0^\theta f(y_{(1)}) y^2 \, dy \\
&= \int_0^\theta \frac{n(\theta - y)^{n-1}}{\theta^n} y^2 \, dy \\
&= \int_0^\theta \frac{1}{\theta^n} [-(\theta - y)^n]' y^2 \, dy \\
&= -\frac{y^2}{\theta^n} (\theta - y)^n |_{y=0}^\theta + 2 \int_0^\theta \frac{y}{\theta^n} (\theta - y)^n \, dy \\
&= 0 + 2 \int_0^\theta \frac{1}{\theta^n} [-\frac{(\theta - y)^{n+1}}{n+1}]' y \, dy \\
&= -2 \frac{1}{\theta^n} \frac{(\theta - y)^{n+1}}{(n+1)} y \Big|_{y=0}^\theta + 2 \int_0^\theta \frac{1}{\theta^n} \frac{(\theta - y)^{n+1}}{n+1} \, dy \\
&= 0 - 2 \frac{1}{\theta^n} \frac{(\theta - y)^{n+2}}{(n+2)(n+1)} \Big|_{y=0}^\theta \\
&= \frac{2\theta^2}{(n+2)(n+1)}
\end{aligned}$$

$$V(Y_{(1)}) = E(Y_{(1)}^2) - [E(Y_{(1)})]^2 = \frac{2\theta^2}{(n+2)(n+1)} - (\frac{\theta}{n+1})^2 = \frac{2\theta^2(n+1) - \theta^2(n+2)}{(n+1)^2(n+2)} = \frac{\theta^2 n}{(n+1)^2(n+2)}$$

Since $\hat{\theta}_1 = (n+1)Y_{(1)} \Rightarrow V(\hat{\theta}_1) = (n+1)^2 V(Y_{(1)}) = \frac{\theta^2 n}{n+2}$.

$$f(y_{(n)}) = \begin{cases} n[\frac{y}{\theta}]^{n-1} \frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
E(y_{(n)}) &= \int_0^\theta \frac{ny^{n-1}}{\theta^n} y \, dy \\
&= \int_0^\theta \frac{ny^n}{\theta^n} \, dy \\
&= \frac{ny^{n+1}}{(n+1)\theta^n} |_{y=0}^{\theta} \\
&= \frac{n\theta^{n+1}}{(n+1)\theta^n} \\
&= \frac{n\theta}{n+1}
\end{aligned}$$

$$\begin{aligned}
E(y_{(n)}^2) &= \int_0^\theta \frac{ny^{n-1}}{\theta^n} y^2 \, dy \\
&= \int_0^\theta \frac{ny^{n+1}}{\theta^n} \, dy \\
&= \frac{ny^{n+2}}{(n+2)\theta^n} |_{y=0}^{\theta} \\
&= \frac{n\theta^2}{n+2}
\end{aligned}$$

$$V(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2 = \frac{n\theta^2}{n+2} - (\frac{n\theta}{n+1})^2 = \frac{n\theta^2(n+1)^2 - n^2\theta^2(n+2)}{(n+1)^2(n+2)} = \frac{\theta^2 n}{(n+1)^2(n+2)}$$

Since $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)} \Rightarrow V(\hat{\theta}_2) = (\frac{n+1}{n})^2 V(Y_{(n)}) = (\frac{n+1}{n})^2 \frac{\theta^2 n}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$. Since both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimator for θ, $MSE(\hat{\theta}_1) = \frac{\theta^2 n}{n+2}$ and $MSE(\hat{\theta}_2) = \frac{\theta^2}{n(n+2)}$,

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)} = \frac{\theta^2}{n(n+2)} \frac{n+2}{\theta^2 n} = \frac{1}{n^2}.$$

9.7
$Y_1 \sim exp(\theta) \Rightarrow E(Y_1) = \theta, V(Y_1) = \theta^2$.
$\hat{\theta}_2 = \bar{Y}$, $E(\bar{Y}) = E(\frac{\sum_{i=1}^n Y_i}{n}) = \frac{\sum_{i=1}^n E(Y_i)}{n} = \frac{nE(Y_1)}{n} = E(Y_1) = \theta$.
Since $Y_i's$ are $i.i.d$, $V(\bar{Y}) = V(\frac{\sum_{i=1}^n Y_i}{n}) = \frac{\sum_{i=1}^n V(Y_i)}{n^2} = \frac{nV(Y_1)}{n^2} = \frac{\theta^2}{n} \Rightarrow MSE(\hat{\theta}_2) = \frac{\theta^2}{n}$.

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)} = \frac{\theta^2}{n} \frac{1}{\theta^2} = \frac{1}{n}.$$