hw5 solutions

9.19

$$f(y) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Notice that $Y \sim Beta(\alpha = \theta, \beta = 1)$, thus

$$E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{\theta}{\theta+1}, \;\; Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\theta}{(\theta+1)^2(\theta+2)}$$

$$E(\bar{Y}) = E(\frac{\sum_{i=1}^n Y_i}{n}) \overset{i.i.d}{=} \frac{\sum_{i=1}^n E(Y_i)}{n} = n\frac{\theta}{\theta+1}/n = \frac{\theta}{\theta+1},$$

$$Var(\bar{Y}) = Var(\frac{\sum_{i=1}^n Y_i}{n}) \overset{i.i.d}{=} \frac{\sum_{i=1}^n Var(Y_i)}{n^2} = n\frac{\theta}{(\theta+1)^2(\theta+2)}/n^2 = \frac{\theta}{(\theta+1)^2(\theta+2)n} \to 0,$$

as $n \to \infty$.

Hence, $MSE(\bar{Y}) = Var(\bar{Y}) \to 0$. By the result in lecture 14 page 2, $\bar{Y}$ is a consistent estimator for $\frac{\theta}{\theta+1}$.

9.20

$Y \sim Bin(n, p)$, thus

$$E(Y) = np, \;\; Var(Y) = np(1-p)$$

$$E(Y/n) = np/n = p, \;\; Var(Y/n) = Var(Y)/n^2 = np(1-p)/n^2 = p(1-p)/n$$

Since $MSE(\bar{Y}) = Var(\bar{Y}) = p(1-p)/n \to 0$ as $n \to \infty$. By the result in lecture 14 page 2, Y/n is a consistent estimator for p.

9.38

a. σ^2 known.

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(y-\mu)^2\}$$

$$L(\mu) = \prod_{i=1}^n f(y_i) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(y_i-\mu)^2\} = \frac{1}{(2\pi)^{n/2}\sigma^n} exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\mu)^2\}.$$

Note that

$$\sum_{i=1}^{n}(y_i-\mu)^2=\sum_{i=1}^{n}(y_i^2-2y_i\mu+\mu^2)=\sum_{i=1}^{n}y_i^2-2\mu\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}\mu^2=\sum_{i=1}^{n}y_i^2-2\mu n\bar{y}+n\mu^2,$$

we get

$$L(\mu)=\frac{1}{(2\pi)^{n/2}\sigma^n}exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\}=\underbrace{\frac{1}{(2\pi)^{n/2}\sigma^n}exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}y_i^2\}}_{h(y_1,\dots,y_n)}\underbrace{exp\{-\frac{1}{2\sigma^2}(-2\mu n\bar{y}+n\mu^2)\}}_{g(\bar{y},\mu)}.$$

By the result in lecture 16 page 3, $\bar{Y}$ is sufficient for μ.

b. μ known.

$$L(\sigma^2)=\prod_{i=1}^{n}f(y_i)=\frac{1}{(2\pi)^{n/2}\sigma^n}exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\}=\underbrace{\frac{1}{(\sigma^2)^{n/2}}exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\}}_{g(\sum_{i=1}^{n}(y_i-\mu)^2,\sigma^2)}\underbrace{\frac{1}{(2\pi)^{n/2}}}_{h(y_1,\dots,y_n)}.$$

By the result in lecture 16 page 3, $\sum_{i=1}^{n}(Y_i-\mu)^2$ is sufficient for σ^2.

9.43

$$L(\alpha)=\prod_{i=1}^{n}f(y_i)=\prod_{i=1}^{n}\frac{\alpha y_i^{\alpha-1}}{\theta^\alpha}=\frac{\alpha^n(\prod_{i=1}^{n}y_i)^{\alpha-1}}{\theta^{\alpha n}}\times 1=\underbrace{\frac{\alpha^n(\prod_{i=1}^{n}y_i)^{\alpha-1}}{\theta^{\alpha n}}}_{g(\prod_{i=1}^{n}y_i,\alpha)}\underbrace{1}_{h(y_1,\dots,y_n)}.$$

By the result in lecture 16 page 3, $\prod_{i=1}^{n}Y_i$ is sufficient for α.

9.58
See 9.34 on page 458. $Y^2\sim exp(\theta)\Rightarrow E(Y^2)=\theta$.

a. See 9.40 on page 463.

$$L(\theta)=\prod_{i=1}^{n}f(y_i)=\prod_{i=1}^{n}\frac{2y_i}{\theta}exp(-y_i^2/\theta)=\frac{2^n\prod_{i=1}^{n}y_i}{\theta^n}exp(-\sum_{i=1}^{n}y_i^2/\theta)=\underbrace{\frac{1}{\theta^n}exp(-\sum_{i=1}^{n}y_i^2/\theta)}_{g(\sum_{i=1}^{n}y_i^2,\theta)}\underbrace{2^n\prod_{i=1}^{n}y_i}_{h(y_1,\dots,y_n)}.$$

By the result in lecture 16 page 3, $\sum_{i=1}^{n}Y_i^2$ is minimal (since unique) sufficient for θ.
b.

$$E(\sum_{i=1}^{n}Y_i^2/n)\overset{i.i.d}{=}\frac{1}{n}nE(Y^2)=\theta.$$

By the procedure in lecture 17 page 4, $\sum_{i=1}^{n} Y_i^2/n$ is MVUE for θ.

9.61

a. See 9.49 on page 463. Let

$$I_{(0,\theta)}(y) = \begin{cases} 1, & 0 \le y \le \theta \\ 0, & \text{otherwise} \end{cases}.$$

We get

$$f(y) = \frac{1}{\theta} I_{(0,\theta)}(y),$$

and

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{(0,\theta)}(y_i) = \underbrace{\frac{1}{\theta^n} \prod_{i=1}^{n} I_{(0,\theta)}(y_{(n)})}_{g(y_{(n)},\theta)} \times \underbrace{1}_{h(y_1,\ldots,y_n)}.$$

By a result in lecture 16 page 3, $Y_{(n)}$ is minimal (since unique) sufficient for θ.

b. See Example 1 on page 446.

$$E(Y_{(n)}) = \frac{n}{n+1}\theta \Rightarrow E(\frac{n+1}{n} Y_{(n)}) = \frac{n+1}{n} \frac{n}{n+1} \theta = \theta.$$

By the procedure in lecture 17 page 4, $\frac{n+1}{n} Y_{(n)}$ is MVUE for θ.

9.63

a.

$$F(y) = \int_0^y \frac{3t^2}{\theta^3} dt = \frac{t^3}{\theta^3}\Big|_0^y = \begin{cases} \frac{y^3}{\theta^3}, & 0 \le y \le \theta \\ 0, & \text{otherwise} \end{cases}.$$

See page 333-334.

$$f(y_{(n)}) = n(F(y))^{n-1} f(y) = n(\frac{y^3}{\theta^3})^{n-1} \frac{3y^2}{\theta^3} = \frac{3ny^{3n-1}}{\theta^{3n}} I_{(0,\theta)}(y).$$

b. See 9.52 on page 464.

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{3y_i^2}{\theta^3} I_{(0,\theta)}(y_i) = \underbrace{3^n \prod_{i=1}^{n} y_i^2}_{h(y_1,\ldots,y_n)} \times \underbrace{\frac{I_{(0,\theta)}(y_{(n)})}{\theta^{3n}}}_{g(y_{(n)},\theta)}.$$

Hence, $Y_{(n)}$ is minimal (since unique) sufficient for θ.

$$E(Y_{(n)}) = \int_0^\theta \frac{3ny^{3n-1}}{\theta^{3n}} y dy = \frac{3ny^{3n+1}}{\theta^{3n}(3n+1)}\Big|_0^\theta = \frac{3n\theta^{3n+1}}{\theta^{3n}(3n+1)} = \theta \frac{3n}{3n+1}.$$

Since $E(\frac{3n+1}{3n} Y_{(n)}) = \frac{3n+1}{3n} \frac{3n+1}{3n} \theta = \theta$, by the procedure in lecture 17 page 4, $\frac{3n+1}{3n} Y_{(n)}$ is MVUE for θ.