hw6 solutions

9.70

$$\mu_1' = E(Y_1) = \lambda \Rightarrow \hat{\lambda} = \underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i}_{\text{first sample moment}} = \bar{Y}$$

9.71

$$\mu_1' = E(Y_1) = 0$$

$$\mu_2' = E(Y_1^2) = Var(Y_1) + (E(Y_1))^2 = \sigma^2 + \mu^2 = \sigma^2 \Rightarrow \hat{\sigma}^2 = \underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i^2}_{\text{second sample moment}} = \overline{Y^2}$$

9.72

$$\mu_1' = E(Y_1) = \mu,\ \ \mu_2' = E(Y_1^2) = Var(Y_1) + (E(Y_1))^2 = \sigma^2 + \mu^2$$

$$\Rightarrow \mu = \mu_1', \sigma^2 = \mu_2' - \mu^2 = \mu_2' - (\mu_1')^2 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y},\ \ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - (\bar{Y})^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2,$$

where the last equality follows from that

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \frac{1}{n}(\sum_{i=1}^{n} Y_i^2 - 2\bar{Y}\sum_{i=1}^{n} Y_i + n(\bar{Y})^2)$$

$$= \frac{1}{n}(\sum_{i=1}^{n} Y_i^2 - 2\bar{Y}n\bar{Y} + n(\bar{Y})^2) = \frac{1}{n}(\sum_{i=1}^{n} Y_i^2 - n(\bar{Y})^2) = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - (\bar{Y})^2$$

9.74

a.

$$E(Y) = \int_0^\theta \frac{2}{\theta^2}(\theta - y)y dy = \int_0^\theta \frac{2}{\theta}y - \frac{2}{\theta^2}y^2 dy$$

$$= \frac{2}{\theta}\frac{y^2}{2}\Big|_0^\theta - \frac{2}{\theta^2}\frac{y^3}{3}\Big|_0^\theta = \theta - \frac{2}{3}\theta = \frac{1}{3}\theta$$

$$\mu_1' = E(Y_1) = \frac{1}{3}\theta \Rightarrow \frac{1}{3}\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y} \Rightarrow \hat{\theta} = 3\bar{Y}$$

b.

$$L(\theta) = 2^n \theta^{-2n} \prod_{i=1}^{n} (\theta - y_i)$$

which can not be factored as a function of $\bar{Y}$, hence $\hat{\theta}$ is not sufficient for θ.

9.80

a.

$$f(y) = \frac{\lambda^y e^{-\lambda}}{y!}, \quad L(\lambda) = \frac{\lambda^{\sum_{i=1}^n y_i} e^{-n\lambda}}{\prod_{i=1}^n y_i!}$$

$$\log L(\lambda) = \log \lambda \sum_{i=1}^{n} y_i - n\lambda - \log \prod_{i=1}^{n} y_i!$$

$$\frac{d \log L(\lambda)}{d\lambda} = \sum_{i=1}^{n} y_i/\lambda - n = 0 \Rightarrow \hat{\lambda} = \sum_{i=1}^{n} y_i/n = \bar{y}$$

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = -\sum_{i=1}^{n} y_i/\lambda^2 < 0$$

Hence, $\hat{\lambda} = \bar{Y}$ maximizes $L(\lambda)$.

b.

$$E(\hat{\lambda}) = E(\sum_{i=1}^{n} y_i/n) = n\lambda/n = \lambda$$

$$Var(\hat{\lambda}) = Var(\sum_{i=1}^{n} y_i/n) = \sum_{i=1}^{n} Var(y_i)/n^2 = n\lambda/n^2 = \lambda/n$$

9.82

a.

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \theta^{-n} 2^n (\prod_{i=1}^{n} y_i)^{2-1} e^{-\sum_{i=1}^n y_i^2/\theta} = \underbrace{\theta^{-n} e^{-\sum_{i=1}^n y_i^2/\theta}}_{g(\sum_{i=1}^n y_i^2, \theta)} \times \underbrace{2^n (\prod_{i=1}^{n} y_i)^{2-1}}_{h(y_1,\ldots,y_n)}.$$

Hence, $\sum_{i=1}^n y_i^2$ is the sufficient statistic for θ by the Factorization Theorem.

b.

$$\log L(\theta) = -n \log \theta + \log(2^n (\prod_{i=1}^{n} y_i)^{2-1}) - \sum_{i=1}^{n} y_i^2/\theta$$

$$\frac{d\log L(\theta)}{d\theta} = -n/\theta + \sum_{i=1}^{n} y_i^2/\theta^2 = 0 \Rightarrow \hat{\theta} = \sum_{i=1}^{n} y_i^2/n$$

$$\left.\frac{d^2\log L(\theta)}{d\theta^2}\right|_{\theta=\hat{\theta}} = n/\theta^2 - 2/\theta^3 \left.\sum_{i=1}^{n} y_i^2\right|_{\theta=\hat{\theta}} = -n/\hat{\theta}^2 < 0$$

Hence, $\hat{\theta}$ maximizes $L(\theta)$.

c.

$$E(Y^2) = \int_0^\infty \frac{1}{\theta} 2y^{2-1} e^{-y^2/\theta} y^2 dy = \int_0^\infty -(e^{-y^2/\theta})' y^2 dy = -y^2 e^{-y^2/\theta}\Big|_0^\infty - \int_0^\infty -e^{-y^2/\theta} 2y^{2-1} dy$$

$$= \int_0^\infty e^{-y^2/\theta} 2y dy = \int_0^\infty -(\theta e^{-y^2/\theta})' dy = -\theta e^{-y^2/\theta}\Big|_0^\infty = \theta$$

$$E(\hat{\theta}) = E(\sum_{i=1}^{n} Y_i^2/n) = E(Y_1^2) = \theta$$

Hence $\hat{\theta}$ is unbiased for θ, and also $\hat{\theta}$ is a function of the sufficient statistic for θ, thus $\hat{\theta}$ is MVUE for θ.

9.83

a.

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = (2\theta + 1)^{-n} I(max_i y_i = y_{(n)} \leq 2\theta + 1)$$

$(2\theta + 1)^{-n}$ is a decreasing function in θ, hence it is maximized by small value of θ. But, $y_{(n)} \leq 2\theta + 1 \Rightarrow \theta \geq \frac{y_{(n)}-1}{2}$, the smallest value θ can take is $\frac{y_{(n)}-1}{2} \Rightarrow \hat{\theta} = \frac{y_{(n)}-1}{2}$ (MLE).

b.

$Var(Y) = \frac{(2\theta+1)^2}{12}$ (see uniform distribution.)

Hence,by the invariance property of MLE, the MLE of $\frac{(2\theta+1)^2}{12}$ is $\frac{(2\hat{\theta}+1)^2}{12} = \frac{y_{(n)}^2}{12}$.