HW #7

$\boxed{10.6/p.485}$

$n = 36$

$H_0: p = 0.5$, $H_A: p \neq 0.5$

Rejection region : $|Y - 18| \geq 4$

(a) See Lecture 22 p.1

$$\alpha = P(\text{reject } H_0 / H_0 \text{ is true}) = P(\underbrace{|Y-18| \geq 4}_{\text{reject } H_0} / \underbrace{p = 0.5}_{H_0 \text{ is true}}) =$$

$$= P(-4 \geq Y - 18 \geq 4 / p = 0.5) = P(-4 + 18 \geq Y \geq 4 + 18 / p = 0.5) =$$

$$= P(14 \geq Y \geq 22 / p = 0.5) = \alpha$$

$Y \sim \text{Binomial}(n = 36, p = 0.5)$

We can use Large Sample Approximation (See Lecture 22 p3)

where $EY = np = 36 \times 0.5 = 18$ and $Var(Y) = np(1-p) = 36 \times 0.5 \times 0.5 =$

$= 9$

$$\Rightarrow \alpha = P(14 \geq Y \geq 22 / p = 0.5) = P\left(\frac{14 - np}{\sqrt{np(1-p)}} \geq \underbrace{\frac{Y - np}{\sqrt{np(1-p)}}}_{\sim N(0,1)} \geq \frac{22 - np}{\sqrt{np(1-p)}} / p = 0.5\right)$$

$$= P\left(\frac{14 - 18}{\sqrt{9}} \geq Z \geq \frac{22 - 18}{\sqrt{9}}\right) = P(-1.\bar{3} \geq Z \geq 1.\bar{3}) = 2 \times P(Z \geq 1.\bar{3}) =$$

$$= 2 \times 0.0918 = 0.1836$$

(b) See Lecture 22 p.2

$$\beta = P(H_A \text{ is rejected} / H_A \text{ is true}) = P(\underbrace{|Y - 18| < 4}_{H_A \text{ rejected}} / \underbrace{p = 0.7}_{H_A \text{ is true}})$$

$$|Y - 18| < 4 \Leftrightarrow |Y - 18| \leq 3 \Leftrightarrow -3 \leq Y - 18 \leq 3 \Leftrightarrow$$

$$\Leftrightarrow 15 \leq Y \leq 21$$

$\beta = P(15 \le Y \le 21 / p = 0.7)$

Again, using the Large Sample Approximation (Lecture 22 p8)

$$\Rightarrow \beta = P\left(\frac{15-np}{\sqrt{np(1-p)}} \le \frac{Y-np}{\sqrt{np(1-p)}} \le \frac{21-np}{\sqrt{np(1-p)}} \;/ p=0.7\right) =$$

$$= P\left(\frac{15-36\times0.7}{\sqrt{36\times0.7\times0.3}} \le Z \le \frac{21-36\times0.7}{\sqrt{36\times0.7\times0.3}}\right) = P(-3.7097 \le Z \le -1.5275)$$

$$= P(Z \le 1.53) - P(Z \le 3.7) \simeq 0.063 - 0 = 0.063$$

Note: The exact values for α and β are:

ⓐ $\alpha = P(14 \ge Y \ge 22 / p = 0.5) = \sum_{y=0}^{14} \binom{36}{y} 0.5^{36} + \sum_{y=22}^{36} \binom{36}{y} 0.5^{36} = 0.243$

ⓑ $\beta = P(15 \le Y \le 21 / p = 0.7) = \sum_{y=15}^{21} \binom{36}{y} 0.7^{y}\, 0.3^{36-y} = 0.092$

$\boxed{10.18 / p.504}$

$var(\hat{\mu}) = 2.5^2$

$n = 40$, $\alpha = 0.01$

$\hat{\mu} = 12.2$

$H_0: \mu = 13.2 \qquad H_A: \mu < 13.2$

— Since our population is normal (with mean 13.2 and st. dev 2.5)

$SE(\hat{\mu}) = \sqrt{\frac{var(\hat{\mu})}{n}}$

$\frac{\hat{\mu} - \mu}{SE(\hat{\mu})}$ is Normal (0,1) distributed

see Lecture 23 p.3

— test statistic: $\hat{\mu}$

rejection region: $\hat{\mu} < \mu + z_{0.01} \times SE(\hat{\mu}) \iff$

$$\hat{\mu} < 13.2 - 2.325 \times \frac{2.5}{\sqrt{40}} = 12.28$$

$\hat{\mu} = 12.2 < 12.28 \Rightarrow$ Reject H_0

10.19 / p. 504

$H_0: \mu = 130$ $\quad H_A: \mu < 130$

$\alpha = 0.05$, $n = 40$, $\bar{Y} = 128.6$ $\quad S = 2.1$

Let's check the conditions from Lecture 23 p.3

$\bar{Y}$ is an estimator for $\hat{\mu}$

$SE(\hat{\mu}) = \sqrt{var(\hat{\mu})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$ $\Rightarrow$ S is an estimator for σ

$\Rightarrow \hat{SE}(\hat{\mu}) = \frac{S}{\sqrt{n}}$

$\frac{\hat{\mu} - \mu}{\hat{SE}(\hat{\mu})} = \frac{\sqrt{n}(\bar{Y} - \mu)}{S}$ approx. Normal $(0,1)$

Thus:

Test Statistic : $\hat{\mu}$

Rejection Region : $\hat{\mu} < \mu + z_{0.05} \times \hat{SE}(\hat{\mu}) \Leftrightarrow$

$$\hat{\mu} < 130 - \frac{1.68\, S}{\sqrt{40}} \Leftrightarrow$$

$$\hat{\mu} < 130 - \frac{1.68 \times 2.1}{\sqrt{40}} = 129.44$$

$$\hat{\mu} = 128.6 < 129.44 \Rightarrow \text{reject } H_0$$

10.20 / p. 504

$H_0: \mu \geq 64$ $\quad H_A: \mu < 64$

$\alpha = 0.01$, $n = 50$, $\bar{Y} = 62$, $S = 8$

See Lecture 23, page 3

$\bar{Y}$ is an estimate for $\hat{\mu}$
s is an estimate for σ , $\hat{SE}(\hat{\mu}) = \frac{s}{\sqrt{n}}$

$$\frac{\hat{\mu}-\mu}{\hat{SE}(\hat{\mu})} = \frac{\sqrt{n}(\bar{Y}-\mu)}{s} \text{ approx. Normal } (0,1)$$

Test statistic: $\hat{\mu}$

Rejection Region $\hat{\mu} < \mu + z_{0.01}\, \hat{SE}(\hat{\mu}) \iff$

$$\hat{\mu} < 64 - 2.325 \cdot \frac{8}{\sqrt{50}} = 61.37$$

$\hat{\mu} = 62 > 61.37 \Rightarrow$ Cannot reject H_0

$\boxed{10.26/\text{p. }506}$

$H_0: p = 0.45$, $H_A: p \neq 0.45$

$n = 80$, $\bar{Y} = \frac{32}{80}$, $\alpha = 0.01$

See Lecture 23 p. 4

$\bar{Y}$ is an estimate for $\hat{p}$
S is an estimate for σ $\quad \hat{SE}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

$$\frac{\hat{p}-p}{\hat{SE}(\hat{p})} = \frac{\sqrt{n}(\bar{Y}-p)}{\sqrt{p(1-p)}} \text{ approx. Normal } (0,1)$$

Test statistic: $\hat{p}$

Rejection Region $\hat{p} < p + z_{0.01/2} \times \hat{SE}(\hat{p})$ or $\hat{p} > p + z_{(1-\frac{0.01}{2})} \times \hat{SE}(\hat{p})$

$$\iff \hat{p} < 0.45 - 2.575\sqrt{\frac{0.45(1-0.45)}{80}} \text{ or } \hat{p} > 0.45 + 2.575\sqrt{\frac{0.45(1-0.45)}{80}}$$

$$\iff \hat{p} < 0.307 \text{ or } \hat{p} > 0.593$$

$\hat{p} = \frac{32}{80} = 0.4 \in (0.307, 0.593) \Rightarrow$ cannot reject H_0

$\boxed{10.37 / p. 510}$

$H_0: \mu = 130$, $H_A: \mu = 128$ see $\boxed{10.19 / p. 506}$

See Lecture 22 p. 2

$$\beta = P(\text{HA rejected} / \text{HA is true}) = P(\underbrace{\bar{Y} > 128.44}_{\text{HA rejected}} / \underbrace{\mu = 128}_{\text{HA true}}) =$$

$$= P\left(\frac{\hat{\mu} - \mu}{\hat{SE}(\hat{\mu})} > \frac{128.44 - \mu}{\hat{SE}(\hat{\mu})} / \mu = 128\right) = P\left(Z > \frac{128.44 - 128}{2.1/\sqrt{40}}\right) =$$

$$= P(Z > 4.337) = 0.0000817$$

$\boxed{10.46 / p. 512}$

$H_0: \theta = \theta_0 \quad H_A: \theta > \theta_0$

Reject if $\dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha$

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha \iff \hat{\theta} - \theta_0 > z_\alpha \times \sigma_{\hat{\theta}} \iff \theta_0 < \hat{\theta} - z_\alpha \times \sigma_{\hat{\theta}}$$

$$\iff \theta_0 < \hat{\theta} + z_{(1-\alpha)} \sigma_{\hat{\theta}}$$

From Lecture 23 p. 2, $\hat{\theta} + z_{(1-\alpha)} \sigma_{\hat{\theta}}$ is the $100(1-\alpha)\%$ lower confidence bound for θ