HW #8

10.52/p.516

(a) $\hat{p}_{0.6} = \frac{55}{70} = 0.786$, $\hat{p}_{0.7} = \frac{23}{70} = 0.329$

$p_{obs} = \hat{p}_{0.6} - \hat{p}_{0.7} = 0.786 - 0.329$

$H_0: \hat{p} = p_{0.6} - p_{0.7} = 0$

$H_A: \hat{p} = p_{0.6} - p_{0.7} > 0$

By Result from Lecture 27, p.3

p-value $= P(p > p_{obs} / H_0 \text{ is true}) = P(p > p_{obs} / \hat{p} = 0) =$

$= P\left(\frac{p - \hat{p}}{S_p} > \frac{p_{obs} - \hat{p}}{S_p} \right) = P\left(Z > \frac{p_{obs} - \hat{p}}{S_p}\right)$

pool the samples to estimate S_p

$S_p = \sqrt{q(1-q)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$ where $q = \frac{55+23}{70+70} = 0.557$

$n_1 = n_2 = 70$

$\Rightarrow S_p = \sqrt{0.557 \times (1 - 0.557)\left(\frac{1}{70} + \frac{1}{70}\right)} = 0.084$

$\Rightarrow$ p-value $= P\left(Z > \frac{p_{obs} - \hat{p}}{S_p}\right) = P\left(Z > \frac{0.786 - 0.329 - 0}{0.084}\right)$

$= P(Z > 5.443) \simeq 0$

(b) $\alpha = 0.05$

See Lecture 27, p.2

p-value $\simeq 0 < 0.05 = \alpha$

$\Rightarrow$ Reject H_0 $\Rightarrow$ the normal cell rate is lower for cells exposed to the higher concentration

$\boxed{10.63/p.526}$

$$\bar{Y} = \frac{\sum_{i=1}^{5} Y_i}{5} = \frac{785+805+790+788+802}{5} = 795$$

$$S = \sqrt{\frac{\sum_{i=1}^{5} (Y_i-\bar{Y})^2}{5-1}} = \sqrt{\frac{(785-795)^2+\dots+(802-795)^2}{4}} = 8.337$$

See Lecture 26 p.2

$H_0: \mu_0 = 800$, $H_A: \mu_0 < 800$

Test Statistic: $\bar{Y}$

Rejection Region: $\bar{Y} < \mu_0 + \frac{t_{n-1,\alpha} \cdot S}{\sqrt{n}}$

$t_{5-1,0.05} = -2.132$ (p.848)

$\Rightarrow$ RR: $795 < 800 - \frac{2.132 \times 8.337}{\sqrt{5}} = 792.051$

$\Rightarrow$ fail to reject H_0

From Lecture 26 p.2 , $Y_1, \dots, Y_5 \sim$ Normal (μ, σ^2)

$\boxed{10.66/p.526\text{-}527}$

$$\bar{Y} = \frac{\sum_{i=1}^{20} Y_i}{20} = \frac{103.768+\dots+90.479}{20} = 89.855$$

$$S = \sqrt{\frac{\sum_{i=1}^{20} (Y_i-\bar{Y})^2}{20-1}} = 14.904$$

See Lecture 26, p.2 Result

$H_0: \mu_0 = 100$, $H_A: \mu_0 < 100$

Test Statistic: $\bar{Y}$

Rejection Region: $\bar{Y} < \mu_0 + \frac{t_{n-1,\alpha} \times S}{\sqrt{n}}$

$t_{20-1,0.01} = -2.539$ (p. 848)

R.R: $89.855 < 100 - \frac{2.539 \times 14.904}{\sqrt{20}} = 91.538$

$\Rightarrow$ Reject Ho

$\boxed{10.67/p.527}$

See Result, Lecture 26, p. 2.

$H_0: \mu_0 = 280$, $H_A: \mu_0 > 280$

Test Statistic: $\bar{Y}$

Rejection Region: $\bar{Y} > \mu_0 + \frac{t_{n-1,1-\alpha} \times S}{\sqrt{n}}$, $t_{10-1,1-0.01} = 2.821$

RR: $358 > 280 + \frac{2.821 \times 54}{\sqrt{10}} = 328.172$ $\Rightarrow$ Reject Ho

$\boxed{10.91/p.547}$

(a)

$H_0: \mu_0 = 7$, $H_A: \mu_0 > 7$

Test Statistic: $\bar{Y}$

Rejection Region: $\bar{Y} > \mu + z_{0.05} \frac{\sigma}{\sqrt{n}}$

$\Rightarrow$ Reject if $\bar{Y} > 7 + 1.645 \frac{\sqrt{5}}{\sqrt{20}} = 7.82$

(b) See Lecture 28 p.1

Power $(\mu_A) = P(H_A \text{ is accepted} / \mu = \mu_A) = P(H_0 \text{ is rejected} / \mu = \mu_A)$

Power $(\mu = 7.5) = P(\bar{Y} > 7.82 / \mu = 7.5) = P\left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} > \frac{7.82 - \mu}{\sigma/\sqrt{n}} / \mu = 7.5\right)$

$= P\left(Z > \frac{7.82 - 7.5}{\sqrt{5}/\sqrt{20}}\right) = P(Z > 0.64) = 0.261$

$$\text{Power}(\mu=8) = P(\bar{Y} > 7.82 / \mu=8) = P\left(Z > \frac{7.82-8}{\sqrt{5/20}}\right) =$$

$$= P(Z > -0.36) = 0.641$$

$$\text{Power}(\mu=8.5) = P(\bar{Y} > 7.82 / \mu=8.5) = P\left(Z > \frac{7.82-8.5}{\sqrt{5/20}}\right) =$$

$$= P(Z > -1.36) = 0.913$$

$$\text{Power}(\mu=9) = P(\bar{Y} > 7.82 / \mu=9) = P\left(Z > \frac{7.82-9}{\sqrt{5/20}}\right) =$$

$$= P(Z > -2.36) = 0.991$$

$\boxed{10.98/p.548}$

See Neyman-Pearson's lemma on lecture 28 p.3

$H_0: \theta=\theta_0$, $H_A: \theta=\theta_A$ (where $\theta_A > \theta_0$)

$\frac{L(\theta_0)}{L(\theta_A)} < k$ RR $\Rightarrow$ UMP test for H_0 vs H_A

$$L(\theta_0) = \frac{1}{\theta_0^n} m^n \prod_{i=1}^{n} y_i^{m-1} \exp\left(-\frac{1}{\theta_0}\sum_{i=1}^{n} y_i^m\right)$$

$$L(\theta_A) = \frac{1}{\theta_A^n} m^n \prod_{i=1}^{n} y_i^{m-1} \exp\left(-\frac{1}{\theta_A}\sum_{i=1}^{n} y_i^m\right)$$

$$\frac{L(\theta_0)}{L(\theta_A)} = \frac{\frac{1}{\theta_0^n} m^n \prod_{i=1}^{n} y_i^{m-1} \exp\left(-\frac{1}{\theta_0}\sum_{i=1}^{n} y_i^m\right)}{\frac{1}{\theta_A^n} m^n \prod_{i=1}^{n} y_i^{m-1} \exp\left(-\frac{1}{\theta_A}\sum_{i=1}^{n} y_i^m\right)} = \left(\frac{\theta_A}{\theta_0}\right)^n \exp\left[\sum_{i=1}^{n} y_i^m \left(\frac{1}{\theta_A}-\frac{1}{\theta_0}\right)\right] < k$$

$$\Leftrightarrow n \ln\left(\frac{\theta_A}{\theta_0}\right) + \sum_{i=1}^{n} y_i^m \left(\frac{1}{\theta_A}-\frac{1}{\theta_0}\right) < \ln k \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{n} y_i^m > \frac{-\ln k + n \ln\left(\frac{\theta_A}{\theta_0}\right)}{\frac{1}{\theta_0}-\frac{1}{\theta_A}} = c$$

$\Leftrightarrow$ RR : $\{\sum_{i=1}^{n} y_i^m > c\}$. choose c such that the size of the test is α

Let $Z = Y^m \Rightarrow Y = Z^{1/m} \Rightarrow y' = \frac{1}{m} z^{\frac{1}{m}-1} = \frac{1}{m} z^{-\frac{m-1}{m}}$

$\Rightarrow f(z) = \left(\frac{1}{\theta}\right) m \left(z^{1/m}\right)^{m-1} \exp(-z/\theta) \cdot \frac{1}{m} z^{-\frac{m-1}{m}} I(z>0)$

$\Rightarrow f(z) = \frac{1}{\theta} e^{-z/\theta} I(z>0) \Rightarrow Z = Y^m \sim \exp(\theta)$

$$\Rightarrow \frac{2\sum_{i=1}^{n} Y_i^m}{\theta_0} \sim \chi^2_{2n}$$

Thus, RR: $\frac{2\sum_{i=1}^{n} Y_i^m}{\theta_0} > \frac{2c}{\theta_0}$ (since RR: $\sum_{i=1}^{n} Y_i^m > c$)

$\Rightarrow$ can use a χ^2_{2n} distribution (and this does not depend on the specific $\theta_A > \theta_0$) $\Rightarrow$ UMP test.

$\boxed{10.100/p. 548}$

See N-P lemma - lecture 28 p. 3

$H_0: \lambda_1 = \lambda_2 = 2$, $H_A: \lambda_1 = \frac{1}{2}$, $\lambda_2 = 3$

$Y_1, \dots Y_n \sim \text{Poisson}(\lambda_1)$

$X_1, \dots X_m \sim \text{Poisson}(\lambda_2)$

$L(\lambda_1) = \frac{\lambda_1^{\sum Y_i} e^{-n\lambda_1}}{\prod^n Y_i!}$, $L(\lambda_2) = \frac{\lambda_2^{\sum^m X_i} e^{-m\lambda_2}}{\prod^m X_i!}$

Since $(Y_1, \dots Y_n)$ independent of $(X_1, \dots X_m)$ $\Rightarrow$

$\Rightarrow L(\lambda_1, \lambda_2) = L(\lambda_1) \cdot L(\lambda_2)$

$$L_0 = L(\lambda_1 = \lambda_2 = 2) = \frac{2^{\sum^n Y_i + \sum^m X_i} \exp(-2n - 2m)}{\prod^n (y_i!) \times \prod^m (x_i!)}$$

$$L_1 = L\left(\lambda_1 = \tfrac{1}{2}, \lambda_2 = 3\right) = \frac{\left(\frac{1}{2}\right)^{\sum^n Y_i} \exp\left(-\frac{n}{2}\right) 3^{\sum^m X_i} \exp(-3m)}{\prod^n (y_i!) \times \prod^m (x_i!)}$$

$$RR: \quad \frac{L_0}{L_1} < k \iff \frac{L(\lambda_1 = \lambda_2 = 2)}{L(\lambda_1 = \frac{1}{2}, \lambda_2 = 3)} < k \iff$$

$$\iff \frac{\dfrac{2^{\sum^{n} y_i + \sum^{m} x_i} \exp(-2n - 2m)}{\prod^{n}(y_i!) \prod^{m}(x_i!)}}{\dfrac{\left(\frac{1}{2}\right)^{\sum^{n} y_i} \cdot 3^{\sum^{m} x_i} \exp\left(-\frac{n}{2} - 3m\right)}{\prod^{n}(y_i!) \prod^{m}(x_i!)}} < k \iff 4^{\sum^{n} y_i} \left(\frac{2}{3}\right)^{\sum^{m} x_i} \exp\left(m - \frac{3}{2}n\right) < k$$

$$\iff \left(\sum^{n} y_i\right) \ln 4 + \left(\sum^{m} x_i\right) \ln \frac{2}{3} < \ln k + \frac{3}{2} n - m = c$$

$$\iff \left(\sum^{n} y_i\right) \ln 4 + \left(\sum^{m} x_i\right) \ln \frac{2}{3} < c$$

Thus $RR: \left(\sum^{n} y_i\right) \ln 4 + \left(\sum^{m} x_i\right) \ln \frac{2}{3} < c$

choose c such that the size of the test is α.