LECTURE - 10

Agenda:

1. Examples

Example: Suppose that the [redacted] lifetime of an electronic component is uniformly distributed over the interval $(0, \theta)$. Suppose that $Y_1, Y_2, \ldots, Y_n$ are lifetimes of a random sample of n [redacted] electronic components. Our task is to provide [redacted] a 95% lower confidence limit for θ. [redacted]

(a) Prove that the distribution function of $Z = \frac{\max(Y_1, Y_2, \ldots, Y_n)}{\theta}$ does not depend on θ.

First note that, the density function of $\overset{Y_{(n)} =}{\max}(Y_1, Y_2, \ldots, Y_n)$ is given by

$$f_{Y_{(n)}}(y) = n\left(F(y)\right)^{n-1} f(y),$$

where F and f are the common distribution function and density function of $Y_1, Y_2, \ldots, Y_n$ respectively. Hence,

$$f_{Y_{(n)}}(y) = \begin{cases} n\left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} & \text{if } 0 \le y \le \theta, \\ 0 & \text{otherwise.} \end{cases}$$

This was proved using the fact that the distribution function of Uniform $(0, \theta)$ is given by

$$F(y) = \begin{cases} \frac{y}{\theta} & \text{if} \quad 0 \leq y \leq \theta, \\ 0 & \text{if} \quad y < 0, \\ 1 & \text{if} \quad y > \theta, \end{cases}$$

and the density function of Uniform $(0, \theta)$ is given by

$$f(y) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \leq y \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

An easy computation gives us that the distribution function of $Y_{(n)}$ is given by

$$F_{Y_{(n)}}(y) = \begin{cases} \frac{y^n}{\theta^n} & \text{if} \quad 0 \leq y \leq \theta, \\ 0 & \text{if} \quad y < 0, \\ 1 & \text{if} \quad y > 0. \end{cases}$$

(How? Just integrate the density function $f_{Y_{(n)}}$)

We are interested in computing F_Z, where $Z = \frac{Y_{(n)}}{\theta}$.

[redacted] Since,

$$F_Z(y) = P(Z \leq y) = P\left(\frac{Y_{(n)}}{6} \leq y\right) = P(Y_{(n)} \leq \theta y),$$

it follows that

$$F_Z(y) = \begin{cases} y^n & \text{if} \quad 0 \leq y \leq 1 \\ 0 & \text{if} \quad y < 0, \\ 1 & \text{if} \quad y > 0. \end{cases}$$

Hence, the distribution function of $Z = \dfrac{\max(Y_1, Y_2, \ldots, Y_n)}{\theta}$

does not depend on θ.

(b) Prove that $P\left(Z \le (0.95)^{1/n}\right) = 0.95$.

By part (a),

$$P\left(Z \le (0.95)^{1/n}\right) = F_Z\left((0.95)^{1/n}\right) = \left((0.95)^{1/n}\right)^n = 0.95$$

(c) Use Z as a pivotal quantity to provide a 95% lower confidence bound for θ.

$$P\left(Z \le (0.95)^{1/n}\right)$$

$$= P\left(\frac{Y_{(n)}}{\theta} \le (0.95)^{1/n}\right)$$

$$= P\left(\underbrace{\frac{Y_{(n)}}{(0.95)^{1/n}}}_{\hat{\theta}_{lower}} \le \theta\right).$$

From part (b), it follows that

$$P(\hat{\theta}_{lower} \le \theta) = 0.95.$$

Hence $\hat{\theta}_{lower} = \dfrac{Y_{(n)}}{(0.95)^{1/n}}$ is a 95% lower confidence bound for θ.