LECTURE - 14

Agenda:

1. Consistency of a statistical estimator
2. Examples
3. LEFTOVER FROM CHAPTER 8: Confidence interval for σ^2.

CONSISTENCY OF A STATISTICAL ESTIMATOR

Let $\hat{\theta}$ be a statistical estimator of a parameter θ obtained from a sample of size n.

Definition: $\hat{\theta}$ is said to be a consistent estimator of θ if, for any positive number ε,

$$\underbrace{\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| \leq \varepsilon) = 1}_{\text{OR } \hat{\theta} \text{ converges to } \theta \text{ in probability}}.$$

Intuitively, what this means is that if the sample size n is large enough, the probability that the estimator $\hat{\theta}$ is very close to the true value θ is almost 1. This is a very desirable quality in a statistical estimator. Any estimator that is not consistent should be viewed suspiciously.

How do we prove that an estimator is consistent? Here are 2 results which are extremely useful.

RESULT 1: If $MSE(\hat{\theta}) \to 0$ as $n \to \infty$, then $\hat{\theta}$ is a consistent estimator of θ.

RESULT 2: Let $\hat{\theta}$ be a consistent estimator of θ, and $\hat{\theta}'$ be a consistent estimator of θ'. Then,

(i) $\hat{\theta} + \hat{\theta}'$ is a consistent estimator of $\theta + \theta'$.
(ii) $\hat{\theta} \times \hat{\theta}'$ is a consistent estimator of $\theta \times \theta'$.
(iii) If $\theta' \neq 0$, then $\dfrac{\hat{\theta}}{\hat{\theta}'}$ is a consistent estimator of θ'.
(iv) If g is a real-valued function that is continuous at θ, then $g(\hat{\theta}_n)$ is a consistent estimator for $g(\theta)$.

Let us look at a few examples.

Example 1: Let $Y_1, Y_2, ---, Y_n$ be a random sample from a population with mean μ and finite variance σ^2. Is $\overline{Y}$ consistent for μ?

Since $MSE(\bar{Y}) = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$,

$\bar{Y}$ is infact a consistent estimator of μ. We noted this fact in STA4321 as the weak law of large numbers.

Example 2: Let $Y_1, Y_2, ---, Y_n$ be a random sample from a population which is Normal(μ, σ^2). Is S^2 a consistent estimator of σ^2?

Since $MSE(S^2) = \frac{2\sigma^4}{n-1} \to 0$ as $n \to \infty$,

S^2 is infact a consistent estimator of σ^2.

Example 3: Let $Y_1, Y_2, ---, Y_n$ be a random sample from a population which is Normal(μ, σ^2). Provide a consistent estimator for $\frac{\mu}{\sigma}$.

Note that,

$\bar{Y}$ is consistent for μ.

S^2 is consistent for $\sigma^2 \Rightarrow \sqrt{S^2} = S$ is consistent for $\sqrt{\sigma^2} = \sigma$ (as $g(x) = \sqrt{x}$ is a continuous function for $x > 0$).

Hence, $\frac{\bar{Y}}{S}$ is a consistent estimator for μ.

LEFTOVER FROM CHAPTER 8: CONFIDENCE INTERVAL FOR σ^2.

Suppose we have a random sample $Y_1, Y_2, \dots, Y_n$ from a population which is Normal(μ, σ^2), μ and σ^2 are both unknown. The goal is to construct a confidence interval for σ^2 with confidence level $1-\alpha$.

Step 1: Note that in Exercise 8.41, we saw that if $\mu = 0$, then $\frac{\sum_{i=1}^{n} Y_i^2}{\sigma^2}$ has a χ^2-distribution

with n degrees of freedom, and hence can be used as a pivotal quantity (The exercise only deals with $n=1$, but the result holds for general n). However, in the current problem, μ is unknown. But we know that

$$\frac{(n-1)S^2}{\sigma^2} \text{ has a } \chi^2\text{-distribution with}$$

$(n-1)$ degrees of freedom. Hence,

$$\frac{(n-1)S^2}{\sigma^2} \text{ is a pivotal quantity.}$$

Step 2: Find a and b such that

$$P\left(a \leq \frac{(n-1)S^2}{\sigma^2} \leq b\right) = 1-\alpha.$$

Note that $\chi^2_{n,1-\alpha}$ denotes the $(1-\alpha)^{th}$ quantile of the χ^2- distribution with n degrees of freedom.

FACT: $$P\left(\chi^2_{n-1,\frac{\alpha}{2}} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1,1-\frac{\alpha}{2}}\right) = 1-\alpha.$$

Hence, $a = \chi^2_{n-1,\frac{\alpha}{2}}$ and $b = \chi^2_{n-1,1-\frac{\alpha}{2}}$.

Step 3: Express $a \leq \frac{(n-1)S^2}{\sigma^2} \leq b$ as $\hat{\sigma}^2_L \leq \sigma^2 \leq \hat{\sigma}^2_U$.

Note that

$$a \leq \frac{(n-1)S^2}{\sigma^2} \leq b$$

$$\Longleftrightarrow \frac{(n-1)S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}.$$

Hence, a $(1-\alpha)$ confidence interval for σ^2 is given by

$$\left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right].$$