LECTURE 17

Agenda:

1. Rao - Blackwell Theorem
2. MVUE Estimator
3. Examples

Consider the case when we have a sample $Y_1, Y_2, \ldots, Y_n$ from a population with an unknown parameter θ.

Let $\hat{\theta}$ be an unbiased estimator of θ, such that $V(\hat{\theta}) < \infty$. Let W be a sufficient estimator of θ. Define the estimator

$$\hat{\theta}^* = E[\hat{\theta} | U].$$

Recall that if X and Y are random variables, then $E[X|Y]$ is defined by $h(Y)$ where

$$h(y) = E[X | Y=y].$$

In short, $E[\hat{\theta}|U]$ is a random variable which is a function of U.

RESULT: (RAO-BLACKWELL THEOREM)

$E[\hat{\theta}^*] = \theta$: $\hat{\theta}^*$ is unbiased for θ.

$V(\hat{\theta}^*) \leq V(\hat{\theta})$: $\hat{\theta}^*$ has a lower variance than $\hat{\theta}$.

Hence, $\hat{\theta}^*$ is an improvement over θ, since it is unbiased, but has a lower variance (which is same as saying it has a lower mean squared error).

MAIN IDEA: If we have an unbiased estimator, we can get an improved version by taking conditional expectation over the sufficient statistic.

HOW DO WE FIND A SUFFICIENT STATISTIC (ESTIMATOR) IN GENERAL?

1. Write down the likelihood $L(y_1, y_2, ---, y_n | \theta)$.
2. Check if we can factorize into the form

$$g(u, \theta) \times h(y_1, y_2, ---, y_n)$$

for some estimator W. Choose W as the sufficient estimator.

This method works out very well for determining sufficient statistics (estimators). For all examples that we will discuss in this course, there will be a unique estimator W which will be sufficient. But often, there are a lot of possible sufficient estimators, and then the question arises, which is the "best" sufficient estimator? We will not go into how to get this "best" sufficient estimator, but just note that such an estimator is called the "minimal sufficient estimator".

HOWEVER, in the examples that we consider, there is a UNIQUE sufficient statistic (estimator).

Hence, that sufficient statistic is indeed MINIMAL SUFFICIENT.

<u>Remark:</u> Of course, the uniqueness above is upto one-to-one transformations, i.e., if U is sufficient, any one-to-one function $h(U)$ is also sufficient.

> **RESULT:** If W is the MINIMAL SUFFICIENT estimator for θ, then any function of W, which is an unbiased estimator for θ, has the smallest variance, in the class of all unbiased estimators

Such an estimator is known as the MINIMUM VARIANCE UNBIASED ESTIMATOR (MVUE) for θ.

The above result gives us a procedure to find the MVUE for θ.

① Use the factorization criterion to get the minimal sufficient estimator W.

② Try to find a function of W, which is unbiased for θ. By the result, this estimator is the MVUE for θ, which means that it has the smallest variance among all unbiased estimators of θ.

<u>Example:</u> Suppose $Y_1, Y_2, ---, Y_n$ is a random sample from a population which is Uniform $(0, \theta)$.

(a) Show that $Y_{(n)} = \max(Y_1, Y_2, ---, Y_n)$ is minimal sufficient for θ.

Note that if X is Uniform $(0, \theta)$ then the density of X is given by

$$f_X(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \le x \le \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Using the indicator notation that we introduced in the previous lecture,

$$f_X(x) = \frac{1}{\theta} 1_{\{0 \le x \le \theta\}}$$

The likelihood of the data at an arbitrary configuration $y_1, y_2, ---, y_n$ is given by

$$L(y_1, y_2, ---, y_n \mid \theta) = f_{Y_1, Y_2, ---, Y_n}(y_1, y_2, ---, y_n)$$

$$= f_{Y_1}(y_1) \times f_{Y_2}(y_2) \times --- \times f_{Y_n}(y_n)$$

$$= \frac{1}{\theta} 1_{\{0 \le y_1 \le \theta\}} \times \frac{1}{\theta} 1_{\{0 \le y_2 \le \theta\}} \times --- \times \frac{1}{\theta} 1_{\{0 \le y_n \le \theta\}}$$

$$= \overbrace{1_{\{0 \le \min(y_1, y_2, ---, y_n)\}}}^{h(y_1, y_2, ---, y_n)} \overbrace{\frac{1_{\{\max(y_1, y_2, ---, y_n) \le \theta\}}}{\theta^n}}^{g(y_{(n)}, \theta)}$$

($\because$ $0 \le y_i \le \theta$ for every $1 \le i \le n$, if and only if $0 \le \min(y_1, y_2, ---, y_n)$ and $\max(y_1, y_2, ---, y_n) \le \theta$)

Hence, by the factorization criterion $Y_{(n)} = \max(Y_1, Y_2, ---, Y_n)$ is sufficient for θ. Since we are only considering examples where there is a unique sufficient estimator, it follows that $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ is the minimal sufficient estimator for θ.

(b) Find a function of $Y_{(n)}$ which is an unbiased estimator of θ.

In one of the previous lectures, we derived that

$$E\left[Y_{(n)}\right] = \left(\frac{n}{n+1}\right)\theta$$

$$\Rightarrow E\left[\left(\frac{n+1}{n}\right)Y_{(n)}\right] = \theta.$$

Hence $\left(\frac{n+1}{n}\right)Y_{(n)}$ is a function of $Y_{(n)}$ which is unbiased for θ.

This mean $\left(\frac{n+1}{n}\right)Y_{(n)}$ is the MVUE for θ.