LECTURE - 18

Agenda:

1. The method of moments.
2. Examples.

The method of moments is the oldest method of finding estimators for a given parameter θ. The idea behind the method of moments is that

" Sample moments are good estimator of the population moments".

Suppose we have a random sample $Y_1, Y_2, \ldots, Y_n$ from a population. Then, the above statement is same as saying

" $\frac{1}{n}\sum_{i=1}^{n} Y_i^k$ (the k^{th} sample moment) is a good

estimator of the parameter $\mu_k' \triangleq E[Y_1^k]$

(the k^{th} population moment)."

GENERAL PROCEDURE FOR THE METHOD OF MOMENTS

① Express the parameter of interest θ, as a function of some population moments.
② Define $\hat{\theta}$ to be the same function of the corresponding sample moments.

Example 1: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population which is Uniform$(\theta, \theta+1)$. Find a method of moments estimator of θ.

① Let us start with the first population moment

$$\mu_1' = E[Y_1] = \frac{\theta+(\theta+1)}{2} = \theta + \frac{1}{2}.$$

Hence, $$\theta = \mu_1' - \frac{1}{2}.$$

② In this step, we substitute corresponding sample moments for population moments. Hence,

$$\hat{\theta} = \underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i}_{\text{First sample moment}} - \frac{1}{2}.$$

Example 2: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population which is Gamma(α, β). Find method of moments estimators for α and β.

(1) Note that

$$\mu_1' = E[Y_1] = \alpha\beta$$
$$\mu_2' = E[Y_1^2] = V(Y_1) + \left(E[Y_1]\right)^2 = \alpha\beta^2 + \alpha^2\beta^2$$

Hence $\mu_1' = \alpha\beta$, and $\mu_2' - (\mu_1')^2 = \alpha\beta^2$

$$\Rightarrow \quad \alpha = \frac{(\mu_1')^2}{\mu_2' - (\mu_1')^2} \quad \text{and} \quad \beta = \frac{\mu_2' - (\mu_1')^2}{\mu_1'}.$$

(2) In this step, we substitute corresponding sample moments for population moments. Hence, $\frac{1}{n}\sum_{i=1}^{n} Y_i$ should be substituted for μ_1' and

$\frac{1}{n}\sum_{i=1}^{n} Y_i^2$ should be substituted for μ_2'.

The method of moments estimators for α and β are given by

$$\hat{\alpha} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2}{\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2} \quad \text{and} \quad \hat{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2}{\frac{1}{n}\sum_{i=1}^{n} Y_i}$$

FACT: The method of moments always gives estimators which are consistent.