LECTURE - 20

Agenda:

Examples : Maximum likelihood estimate (MLE).

Example 1: Suppose $Y_1, Y_2, ---, Y_n$ is a random sample from a population which is Exponential (θ). Provide the Maximum likelihood estimate of θ.

Step 1: Write down the likelihood at $y_1, y_2, ---, y_n$

$$L(y_1, y_2, ---, y_n \mid \theta) = f_{Y_1, Y_2, ---, Y_n}(y_1, y_2, ---, y_n)$$

$$= \prod_{i=1}^{n} f_{Y_i}(y_i) \quad (\because \text{By independence})$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-y_i/\theta}$$

$$= \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} y_i}{\theta}}.$$

Step 2: Find the value of θ which maximizes the likelihood.

We want to maximize the function

$L(\theta)$ (shorthand for $L(y_1, y_2, \ldots, y_n \mid \theta)$).

<u>SIMPLIFYING RULE:</u> If $L(\theta)$ contains an exponential term, it is often useful to try and maximize $\log L(\theta)$ instead of $L(\theta)$. Note that if $\hat{\theta}$ maximizes $\log L(\theta)$, it also maximizes $L(\theta)$.

So let us maximize

$$\log L(\theta) = \log\left(\frac{1}{\theta^n} e^{-\sum_{i=1}^n y_i/\theta}\right) = -n\log\theta - \frac{\sum_{i=1}^n y_i}{\theta}.$$

Using calculus, let us solve $\frac{\partial}{\partial\theta}\log L(\theta) = 0$.

$$\frac{d}{d\theta}\log L(\theta) = 0$$

$$\Leftrightarrow \quad \frac{d}{d\theta}\left(-n\log\theta - \frac{\sum_{i=1}^n y_i}{\theta}\right) = 0$$

$$\Leftrightarrow \quad -\frac{n}{\theta} + \frac{\sum_{i=1}^n y_i}{\theta^2} = 0$$

$$\Leftrightarrow \quad \theta = \sum_{i=1}^n y_i / n.$$

It is easy to check that

$$\frac{\partial^2}{\partial \theta^2} \log L(\theta) \Big|_{\theta = \frac{\sum_{i=1}^n y_i}{n}} = - \frac{n^3}{\left(\sum_{i=1}^n y_i\right)^2} < 0.$$

Hence $\log L(\theta)$ is maximized at $\theta = \frac{\sum_{i=1}^n y_i}{n}$. The MLE is $\boxed{\hat{\theta} = \frac{\sum_{i=1}^n Y_i}{n}}$.

<u>Example 2:</u> Suppose $Y_1, Y_2, ---, Y_n$ is a random sample from a population which is Uniform$(0, \theta)$. Provide the maximum likelihood estimate of θ.

<u>Step 1:</u> Write down the likelihood at an arbitrary possible configuration $y_1, y_2, ---, y_n$.

$$L(y_1, y_2, ---, y_n) = f_{Y_1, Y_2, ---, Y_n}(y_1, y_2, ---, y_n)$$

$$= \prod_{i=1}^n f_{Y_i}(y_i)$$

$$= \prod_{i=1}^n \frac{1}{\theta} 1_{\{0 \le y_i \le \theta\}}$$

$$= \frac{1}{\theta^n} \prod_{i=1}^n 1_{\{0 \le y_i \le \theta\}}$$

$$= \frac{1}{\theta^n} \, 1_{\{0 \le y_i \le \theta \text{ for every } i = 1,2,\dots,n\}}$$

$$= \frac{1}{\theta^n} \, 1_{\{\max(y_1, y_2, \dots, y_n) \le \theta, \; \min(y_1, y_2, \dots, y_n) \ge 0\}}$$

$$= \frac{1}{\theta^n} \, 1_{\{\max(y_1, y_2, \dots, y_n) \le \theta\}} \, 1_{\{\min(y_1, y_2, \dots, y_n) \ge 0\}}$$

<u>Step 2:</u> Find the value of θ which maximizes the likelihood.

$$L(\theta) = \frac{1}{\theta^n} \, 1_{\{\max(y_1, y_2, \dots, y_n) \le \theta\}} \, 1_{\{\min(y_1, y_2, \dots, y_n) \ge 0\}}$$

Note that $L(\theta) = 0$ if $\theta < \max(y_1, y_2, \dots, y_n)$. Hence the value of θ maximizing $L(\theta)$ must lie in the interval $[\max(y_1, y_2, \dots, y_n), \infty)$, since $L(\theta)$ is a non-negative function. On this interval, $L(\theta) = \frac{1}{\theta^n} 1_{\{\min(y_1, y_2, \dots, y_n) \ge 0\}}$.
The function $\frac{1}{\theta^n}$ is a decreasing function on $[\max(y_1, y_2, \dots, y_n), \infty)$. Hence, $L(\theta)$ is maximized at $\theta = \max(y_1, y_2, \dots, y_n)$.

The MLE is $\boxed{\hat{\theta} = \max(Y_1, Y_2, \dots, Y_n)}$.