

## LECTURE - (22)

### Agenda:

- ① Level of a statistical test.
- ② Statistical tests using large sample approximations.

Recall from the previous lecture, that in a statistical hypothesis testing problem, we need to decide between a "well-established" NULL HYPOTHESIS ( $H_0$ ) and a "challenger" ALTERNATIVE HYPOTHESIS ( $H_A$ ), based on given data. A statistical testing procedure solves this problem by first constructing an appropriate TEST STATISTIC from the data, and then using the test statistic to construct a REJECTION REGION. If the observed value of the test statistic lies in the rejection region, we reject  $H_0$ .

~~There~~ There can be several testing procedures for a given problem, and we need some measures to evaluate the quality of statistical testing procedure. The first measure is defined as

$$\alpha \triangleq P(\text{Reject } H_0 \mid H_0 \text{ is true}).$$

$\alpha$  is known as the "PROBABILITY OF TYPE I ERROR", or sometimes as the "LEVEL" of the statistical testing procedure.

The second measure is defined as

$$\beta \triangleq P(H_A \text{ is rejected} \mid H_A \text{ is true}).$$

$\beta$  is known as the "PROBABILITY OF TYPE II ERROR", and  $1 - \beta$  is known as the "POWER" of the statistical testing procedure.

TYPICALLY, WHILE CONSTRUCTING TESTING PROCEDURES, WE PRESPECIFY <sup>OR FIX</sup> THE LEVEL  $\alpha$  (mostly  $\alpha = 5\%$ ). SO THE CHALLENGE IS TO FIND A TEST WITH THE LARGEST POSSIBLE "POWER" FOR A FIXED LEVEL  $\alpha$ .

In this course, however, we will just look at the standard testing procedures that are employed in ~~some~~ natural situations arising in real-life applications. We will not worry about how good or bad these procedures are in terms of ~~some~~ POWER.

## STATISTICAL TESTS USING LARGE SAMPLE APPROXIMATIONS

Suppose we have a population with an unknown parameter  $\theta$ .

$$H_0: \theta = \theta_0 \quad \text{v.s.} \quad H_A: \theta > \theta_0.$$

Data:  $Y_1, Y_2, \dots, Y_n$

ASSUMPTION: It is possible to construct an estimator  $\hat{\theta}$  for  $\theta$  such that for large enough  $n$ ,

$$Z = \frac{\hat{\theta} - \theta}{SD(\hat{\theta})} \text{ is approximately Normal}(0,1).$$

TEST STATISTIC:  $\hat{\theta}$

If  $\hat{\theta}$  is large, we can expect the true  $\theta$  to be large and hence the alternative hypothesis  $H_A$  to be true.

REJECTION REGION:  $\{\hat{\theta} > k\}$ .

CHOICE OF  $k$ : As previously stated, we often specify the level  $\alpha$  of the test, i.e.,  $k$  should be chosen so that  $P(H_0 \text{ is rejected} | H_0 \text{ is true}) = \alpha$ , i.e.,

$$P(\hat{\theta} > k \mid \theta = \theta_0) = \alpha.$$

Note that

$$P(\hat{\theta} > k \mid \theta = \theta_0) = \alpha$$

$$\Rightarrow P\left(\frac{\hat{\theta} - \theta_0}{SD(\hat{\theta})} > \frac{k - \theta_0}{SD(\hat{\theta})} \mid \theta = \theta_0\right) = \alpha$$

$$\Rightarrow P\left(Z > \frac{k - \theta_0}{SD(\hat{\theta})}\right) = \alpha, \text{ where } Z \text{ has}$$

a Normal(0,1) distribution.

$$\Rightarrow \frac{k - \theta_0}{SD(\hat{\theta})} = z_{1-\alpha}, \text{ where } z_{1-\alpha} \text{ denotes}$$

the  $(1-\alpha)^{\text{th}}$  ~~quantile~~ quantile of the Normal(0,1)

distribution. Hence  $k = \theta_0 + z_{1-\alpha} SD(\hat{\theta})$ .

The rejection region is given by

$$\{\hat{\theta} > \theta_0 + z_{1-\alpha} SD(\hat{\theta})\}$$

FACT: OFTEN  $SD(\hat{\theta})$  IS A FUNCTION OF SOME UNKNOWN POPULATION PARAMETERS, AND IS REPLACED

~~xxx~~ and  $\sigma^2$  denote the variance of the sales of the salespeople in the whole corporation.

BY AN APPROPRIATE ESTIMATE  $\hat{SD}(\bar{Y})$ .

Example: A vice president for a sales claims that he has increased the average sales of the salespeople to more than 15 per week. Suppose 36 salespeople are selected at random, and their number of sales is recorded for one week. The mean and variance of these sales were 17 and 9 respectively. Does the evidence support the vice-president's claim? Use a test with level  $\alpha = 0.05$ .

NOTE: This problem appears in the book, but I have changed it.

Let  $Y_1, Y_2, \dots, Y_{36}$  be sales of the 36 salespeople.

Let  $\mu$  denote the average sales of the salespeople in the whole corporation. Then, the hypothesis testing problem is

$$H_0: \mu = 15 \quad \text{v.s.} \quad H_A: \mu > 15.$$

Note that  $\bar{Y} = \frac{1}{36} \sum_{i=1}^{36} Y_i$  is an estimator for  $\mu$ .

We also know that for large enough sample size  $n$ ,

$$\frac{\bar{Y}_n - \mu}{SD(\bar{Y}_n)} = \frac{\bar{Y}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma}$$

has an approximate Normal  $(0,1)$  distribution  
 Hence, the main assumption required to construct the statistical test based on the large sample approximation procedure is satisfied.

TEST STATISTIC :  $\bar{Y}$

REJECTION REGION :  $\{\bar{Y} > k\}$ .

Note that, in general,  $k = \theta_0 + z_{1-\alpha} SD(\bar{\theta})$ .

Here,

$$\theta_0 = 15, \alpha = 0.05 \text{ and } SD(\bar{\theta}) = \frac{\sigma}{\sqrt{n}}$$

Since  $\sigma$  is unknown, we replace it by a natural estimate  $S$ . Hence,

$$k = 15 + z_{0.95} \frac{S}{\sqrt{36}} = 15 + \frac{1.6455 \cdot 3}{6}$$

For the observed sample,  $\bar{Y} = 17, S^2 = 9$ .

~~Since~~ Since  $17 > 15 + \frac{1.645 \cdot 3}{6}$ , ~~we reject  $H_0$~~

~~we~~ we ~~reject  $H_0$~~  accept  $H_1$ .