LECTURE – 24

<u>Agenda:</u>

1. Calculating Type II Error probabilities
2. Finding an appropriate sample size

Consider the familiar situation where we have data from a population with unknown parameter θ. Suppose we want to test the one-sided hypothesis

$$H_0: \theta = \theta_0 \quad \text{v.s.} \quad H_A: \theta > \theta_0 .$$

We know that a level-α testing procedure is given by

TEST STATISTIC: $\hat{\theta}$

REJECTION REGION: $\hat{\theta} > \theta_0 + z_{1-\alpha} \widehat{SE}(\hat{\theta})$,

where the assumption is that $\hat{\theta}$ is an estimator of θ satisfying $\dfrac{\hat{\theta} - \theta_0}{\widehat{SE}(\hat{\theta})}$ is approximately Normal$(0,1)$. Recall that the Type II Error probabilities for a test are defined by

$$\beta(\theta_A) = P(\, H_A \text{ is rejected} \mid \theta = \theta_A)$$

for every $\theta_A \in H_A$, In our specific case, $H_A = \{\theta : \theta > \theta_0\}$. For any $\theta_A \in H_A$,

$$
\begin{aligned}
\beta(\theta_A) &= P\{\, H_A \text{ is rejected} \mid \theta = \theta_A \,\} \\
&= P\left(\hat{\theta} < \theta_0 + z_{1-\alpha}\, \hat{SE}(\hat{\theta}) \,\middle|\, \theta = \theta_A \right) \\
&= P\left(\frac{\hat{\theta} - \theta_A}{\hat{SE}(\hat{\theta})} < \frac{\theta_0 - \theta_A}{\hat{SE}(\hat{\theta})} + z_{1-\alpha} \,\middle|\, \theta = \theta_A \right) \\
&= P\left(\text{Normal}(0,1) < \frac{\theta_0 - \theta_A}{\hat{SE}(\hat{\theta})} + z_{1-\alpha} \right). \\
&= \Phi\left(\frac{\theta_0 - \theta_A}{\hat{SE}(\hat{\theta})} + z_{1-\alpha} \right)
\end{aligned}
$$

(Recall that $\Phi(z) = P(Z \le z)$ where Z is Normal(0,1))

RESULT: For the standard level-α testing procedure for $\boxed{H_0 : \theta = \theta_0 \text{ v.s. } H_A : \theta > \theta_0}$, the type II Error probabilities are given by

$$
\beta(\theta_A) = \Phi\left(\frac{\theta_0 - \theta_A}{\hat{SE}(\hat{\theta})} + z_{1-\alpha} \right)
$$

for every $\theta_A > \theta_0$.

We can derive the following result for testing $H_0: \theta = \theta_0$ v.s. $H_A: \theta < \theta_0$.

RESULT: For the standard level-α testing procedure for $\boxed{H_0: \theta = \theta_0 \quad \text{v.s.} \quad H_A: \theta < \theta_0}$, the Type II Error probabilities are given by

$$\beta(\theta_A) = 1 - \Phi\left(\frac{\theta_0 - \theta_A}{\hat{SE}(\hat{\theta})} + z_\alpha \right)$$

for every $\theta_A < \theta_0$.

Example: Suppose that the vice president of a company wants to verify a claim by an analyst that the average number of customer calls per week (denoted by μ) has increased from the long-time average of 18. He obtains a sample of 36 weeks, and the average customer calls in these 36 weeks turns out to be 20. Suppose the standard deviation of the sample is 5. If the standard level-0.05 test is used for this problem, find the type II Error probability at the alternative $\mu_A = 20$.

$H_0: \mu = 18$ v.s. $H_A: \mu > 18$.

Let $Y_1, Y_2, ---, Y_{36}$ be the average calls in each of the 36 sampled weeks. Then, we know that

$$\frac{(\bar{Y} - \mu)}{S/\sqrt{36}} \text{ is approximately Normal}(0,1).$$

Hence, the level-0.05 testing procedure is given by

TEST STATISTIC: $\hat{\mu} = \bar{Y}$.

REJECTION REGION: $\hat{\mu} > \mu_0 + z_{1-\alpha} \widehat{SE}(\hat{\mu})$, which is the same as

$$\bar{Y} > 18 + \frac{1.68\, S}{6}$$

We want the Type II Error probability at $\mu_A = 20$. It is given that $S = 5$. Using the result derived in this lecture,

$$\beta(20) = \Phi\left(\frac{18 - 20}{5/6} + 1.68 \right)$$

$$= \Phi\left(1.68 - \frac{12}{5} \right)$$

$$= \Phi(-0.72).$$

Suppose we have a random sample from a population with mean μ. We want to test the hypotheses

$$H_0: \mu = \mu_0 \quad \text{v.s.} \quad \underbrace{H_A: \mu > \mu_0.}$$

Known as the UPPER TAIL ONE-SIDED ALTERNATIVE.

Our client tells us that he definitely wants the level of the test to be α and the Type II Error probability at a particular alternative μ_A to be β.
WHAT IS THE REQUIRED SAMPLE SIZE NEEDED IN ORDER TO SATISFY THESE REQUIREMENTS?

RESULT: The required minimum sample size to obtain a level-α testing procedure such that $\beta(\mu_A) = \beta$ is given by

$$\boxed{n = \frac{(z_{1-\alpha} + z_{1-\beta})^2 S^2}{(\mu_A - \mu_0)^2}}$$

THE SAME FORMULA HOLDS IF WE WANT TO TEST $H_0: \mu = \mu_0$ v.s. $\underbrace{H_A: \mu < \mu_0.}$

Known as the LOWER TAIL ONE-SIDED ALTERNATIVE.

In the example regarding the average number of calls, the required sample size to obtain a testing procedure with level = 0.05 and $\beta(20) = 0.01$ is given by

$$n = \frac{\left(z_{1-0.05} + z_{1-0.01}\right)^2 S^2}{(18-20)^2}$$

$$= \frac{(1.68 + 2.3)^2 \, 25}{4}$$

$$\simeq \underline{100}$$