LECTURE - 25

Agenda:

1. Relationship between hypothesis testing procedures and confidence intervals

Consider the standard situation for hypothesis testing that we have assumed in the previous lectures. We have data from a population with an unknown parameter θ. Let us assume we have an estimator $\hat{\theta}$ of θ such that

$$\frac{\hat{\theta} - \theta}{\widehat{SE}(\hat{\theta})} \text{ is approximately Normal}(0,1).$$

Suppose we want to test the two hypotheses

$$H_0 : \theta = \theta_0 \quad \text{v.s.} \quad H_A : \theta \neq \theta_0 .$$

We derived the following testing procedure with level α.

TEST STATISTIC : $\hat{\theta}$

REJECTION REGION: $\hat{\theta} < \theta_0 + z_{\alpha/2}\widehat{SE}(\hat{\theta})$ or

$\hat{\theta} > \theta_0 + z_{1-\frac{\alpha}{2}}\widehat{SE}(\hat{\theta})$.

Alternatively, we accept H_0 if

$$\theta_0 + z_{\alpha/2}\hat{SE}(\hat\theta) < \hat\theta < \theta_0 + z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta)$$

$$\Leftrightarrow \quad \hat\theta + z_{\alpha/2}\hat{SE}(\hat\theta) < \theta_0 < \hat\theta + z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta)$$

$$\Leftrightarrow \quad \hat\theta - z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta) < \theta_0 < \hat\theta + z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta) \quad \left(\because z_{\alpha/2} = -z_{1-\frac{\alpha}{2}}\right)$$

$$\Leftrightarrow \quad \theta_0 \in \left[\hat\theta - z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta), \hat\theta + z_{1-\frac{\alpha}{2}}\hat{SE}(\hat\theta)\right]. \qquad \text{--- } (1)$$

Now, let us consider the problem of deriving a $(1-\alpha)$ - confidence interval for θ. Since

$$g(\text{Data}, \theta) = \frac{\hat\theta - \theta}{\hat{SE}(\hat\theta)} \text{ is approximately}$$

Normal(0, 1), it can be used as a pivotal quantity.

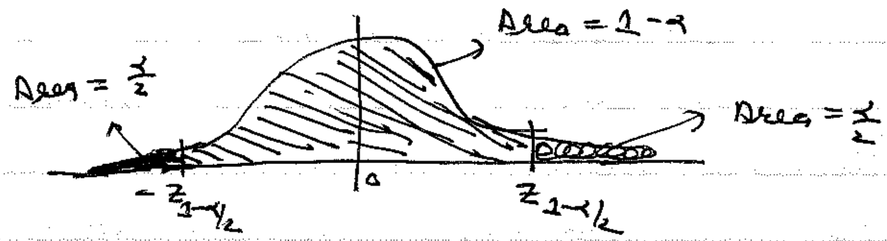
Step 1: Use $\dfrac{\hat\theta - \theta}{\hat{SE}(\hat\theta)}$ as a pivotal quantity

Step 2: Find a, b such that

$$P\left(a \le \frac{\hat\theta - \theta}{\hat{SE}(\hat\theta)} \le b\right) = 1-\alpha.$$

Note that

$$P\left(-z_{1-\frac{\alpha}{2}} \le \text{Normal}(0,1) \le z_{1-\frac{\alpha}{2}} \right) = 1-\alpha$$



It follows that $a = -z_{1-\frac{\alpha}{2}}$ and $b = z_{1-\frac{\alpha}{2}}$.

Step 3: Convert $a \le \dfrac{\hat{\theta} - \theta}{\hat{SE}(\hat{\theta})} \le b$ to the form

$\hat{\theta}_L \le \theta \le \hat{\theta}_U$.

Note that

$$-z_{1-\frac{\alpha}{2}} \le \frac{\hat{\theta} - \theta}{\hat{SE}(\hat{\theta})} \le z_{1-\frac{\alpha}{2}}$$

$$\Leftrightarrow \hat{\theta} - z_{1-\frac{\alpha}{2}} \hat{SE}(\hat{\theta}) \le \theta \le \hat{\theta} + z_{1-\frac{\alpha}{2}} \hat{SE}(\hat{\theta})$$

$$\Leftrightarrow \theta \in \left[\hat{\theta} - z_{1-\frac{\alpha}{2}} \hat{SE}(\hat{\theta}),\ \hat{\theta} + z_{1-\frac{\alpha}{2}} \hat{SE}(\hat{\theta}) \right] \quad \text{— (2)}$$

Hence, comparing (1) and (2), we obtain that the testing procedure for $H_0: \theta = \theta_0$ v.s. $H_1: \theta \neq \theta_0$ accepts H_0 if θ_0 lies in the $(1-\alpha)$-confidence interval for θ.

We established above a connection between the hypothesis testing procedure for a two-sided alternative hypothesis $H_a: \theta \neq \theta_0$ and the two-sided confidence interval for θ. A similar connection can be established between one-sided alternative hypotheses and corresponding one-sided confidence intervals. See Homework 7, Problem 10.46.