LECTURE - 26

Agenda:

- Small-sample hypothesis testing for μ and $\mu_1 - \mu_2$.

Consider the following situation. Suppose $Y_1, Y_2, ---, Y_n$ is a random sample from a population which is normal with mean μ. We wish to test the hypotheses

$$H_0 : \mu = \mu_0 \quad \text{v.s.} \quad H_A : \mu > \mu_0$$

with level α. HOWEVER, THE SAMPLE SIZE n IS NOT LARGE ENOUGH TO APPROXIMATE THE DISTRIBUTION OF $\frac{\sqrt{n}(\bar{Y}-\mu)}{S}$ BY THE NORMAL $(0,1)$ DISTRIBUTION.

But this does not pose a problem, since the assumption that the population is normal implies that

$\frac{\sqrt{n}(\bar{Y}-\mu)}{S}$ has a t-distribution with $n-1$ degrees of freedom.

Recall that $t_{n-1,1-\alpha}$ denotes the $(1-\alpha)^{th}$ quantile of the t- distribution with $n-1$ degrees of freedom.

RESULT: If $Y_1, Y_2, \ldots, Y_n$ is Normal(μ, σ^2) with μ, σ^2 both unknown, then the level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu > \mu_0$ is given by

TEST STATISTIC: $\overline{Y}$

REJECTION REGION: $\overline{Y} > \mu_0 + \dfrac{t_{n-1,1-\alpha} S}{\sqrt{n}}$

This testing procedure is known as the exact or small-sample testing procedure for μ.

RESULT: If $Y_1, Y_2, \ldots, Y_n$ is Normal(μ, σ^2) with μ, σ^2 both unknown, then the level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu < \mu_0$ is given by

TEST STATISTIC: $\overline{Y}$

REJECTION REGION: $\overline{Y} < \mu_0 + \dfrac{t_{n-1,\alpha} S}{\sqrt{n}}$

RESULT: If $Y_1, Y_2, \ldots, Y_n$ is Normal(μ, σ^2) with μ, σ^2 both unknown, then the level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu \neq \mu_0$ is given by

TEST STATISTIC: $\bar{Y}$

REJECTION REGION: $\bar{Y} < \mu_0 + \dfrac{t_{n-1,\frac{\alpha}{2}} S}{\sqrt{n}}$ or

$$\bar{Y} > \mu_0 + \frac{t_{n-1,1-\frac{\alpha}{2}} S}{\sqrt{n}}.$$

As you can see, the only difference between the large sample test and exact test for μ is that the Normal z quantiles are replaced by the corresponding t quantiles.

Consider the following situation. Suppose we have random samples from two normal populations: $Y_{11}, Y_{12}, \ldots, Y_{1n_1}$ is a random sample from a Normal(μ_1, σ^2) population, and $Y_{21}, Y_{22}, \ldots, Y_{2n_2}$ is a random sample from a Normal(μ_2, σ^2) population.

We want to test three different sets of hypotheses.

1. $H_0: \mu_1 - \mu_2 = D_0$ v.s. $H_A: \mu_1 - \mu_2 > D_0$.
2. $H_0: \mu_1 - \mu_2 = D_0$ v.s. $H_A: \mu_1 - \mu_2 < D_0$.
3. $H_0: \mu_1 - \mu_2 = D_0$ v.s. $H_A: \mu_1 - \mu_2 \neq D_0$.

Recall that $\dfrac{(\overline{Y}_1 - \overline{Y}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ has a

t-distribution with n_1+n_2-2 degrees of freedom, where $S_p^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$ is the pooled sample variance.

RESULT: Suppose $Y_{11}, Y_{12}, ---, Y_{1n_1}$ and $Y_{21}, Y_{22}, ---, Y_{2n_2}$ are independent random samples from a Normal(μ_1, σ^2) and Normal(μ_2, σ^2) population respectively.

(1) The level-α testing procedure for testing $H_0: \mu_1-\mu_2 = D_0$ v.s. $H_A: \mu_1-\mu_2 > D_0$ is given by

TEST STATISTIC: $\overline{Y}_1 - \overline{Y}_2$

REJECTION REGION:

$$\overline{Y}_1 - \overline{Y}_2 > D_0 + \underline{t_{n_1+n_2-2,\, 1-\alpha}}\, S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

(2) The level-α testing procedure for testing $H_0: \mu_1-\mu_2 = D_0$ v.s. $H_A: \mu_1-\mu_2 < D_0$ is given by

TEST STATISTIC: $\overline{Y}_1 - \overline{Y}_2$

REJECTION REGION:

$$\overline{Y}_1 - \overline{Y}_2 < D_0 + t_{n_1+n_2-2,\, \alpha}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

③ The level-α testing procedure for testing $H_0: \mu_1 - \mu_2 = D_0$ v.s. $H_A: \mu_1 - \mu_2 \neq D_0$ is given by

TEST STATISTIC: $\overline{Y}_1 - \overline{Y}_2$

REJECTION REGION:

$$\overline{Y}_1 - \overline{Y}_2 < D_0 + t_{n_1+n_2-2,\, \frac{\alpha}{2}}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad \text{or}$$

$$\overline{Y}_1 - \overline{Y}_2 > D_0 + t_{n_1+n_2-2,\, 1-\frac{\alpha}{2}}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}.$$