LECTURE-28

Agenda:

1. Recap: Power of a test
2. Simple and Composite hypotheses
3. Neyman-Pearson lemma and the most powerful test

Recall that for testing H_0 v.s. H_A, which are two hypotheses concerning an unknown parameter θ, we had defined the "power" of an testing procedure at a specific alternative θ_A as

$$\boxed{\begin{aligned} \text{Power}(\theta_A) &= 1 - \underbrace{\beta(\theta_A)}_{\text{Type II Error probability at } \theta_A} \\ &= 1 - P(H_A \text{ is rejected} \mid \theta = \theta_A) \\ &= P(H_A \text{ is accepted} \mid \theta = \theta_A) \end{aligned}}$$

Hence, the power at the alternative θ_A is the probability of correctly accepting H_A when the true value of θ is θ_A.

SO, WE WOULD LIKE THE POWER TO BE AS LARGE AS POSSIBLE.

GENERAL QUESTION: For a general testing problem, can we find a testing procedure which has the highest power (assuming the level is fixed at, say, α)?

ANSWER: In various situations, we can.

Before we proceed, let us define the concepts of SIMPLE and COMPOSITE hypotheses.

DEFINITION: Suppose we have data from a population with unknown parameter θ. A hypothesis is said to be SIMPLE if it uniquely specifies the distribution of the population from which the sample is taken. [IN MOST CASES, THIS IS SAME AS SAYING, THAT THE HYPOTHESES UNIQUELY IDENTIFIES THE VALUE OF θ].

(BUT NOT ALWAYS)

Any hypothesis that is not a SIMPLE hypothesis is called a COMPOSITE hypothesis.

The Neyman-Pearson Lemma provides a method to obtain the MOST POWERFUL test, among all tests with level-α, if both H_0 and H_A are simple hypotheses.

THE NEYMAN – PEARSON LEMMA: Suppose that we wish to test the simple null hypothesis $H_0: \theta = \theta_0$ versus the simple alternative hypothesis $H_A: \theta = \theta_A$, based on a random sample $Y_1, Y_2, \ldots, Y_n$ from a distribution with parameter θ. Recall that $L(\theta)$ denotes the likelihood of the sample when the value of the parameter is θ. Then, the test that maximizes the power at θ_A has a rejection region determined by

$$\frac{L(\theta_0)}{L(\theta_A)} < k.$$

The value of k is chosen so that the level is equal to α. Such a test is called the MOST POWERFUL LEVEL-α TEST FOR H_0 v.s. H_A.

Example: Suppose Y is a sample of size 1 from a population with density

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & \text{if } 0<y<1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the most powerful test with level $\alpha = 0.05$ for testing $H_0: \theta = 2$ v.s. $H_A: \theta = 1$.

Note that both H_0 and H_A are simple, as specifying the value of θ, completely specifies the density of the population. Hence, by the Neyman-Pearson lemma, the MOST POWERFUL test for H_0 v.s. H_A has a rejection region of the form

$$\frac{L(\theta_0)}{L(\theta_A)} < k,$$ where k is chosen so that level $= 0.05$.

Note that $L(\theta_0) = f(y \mid \theta = 2) = 2y$, and $L(\theta_A) = f(y \mid \theta = 1) = 1$, for $0 < y < 1$. Hence, the rejection region is of the form

$$2y < k \quad \text{or} \quad y < \frac{k}{2}.$$

Let us calculate the value of k.

$$\text{Level} = 0.05 \Rightarrow P\left(Y < \frac{k}{2} \,\middle|\, \theta = 2 \right) = 0.05$$

$$\Rightarrow \int_0^{k/2} 2y\,dy = 0.05$$

$$\Rightarrow [y^2]_0^{k/2} = 0.05$$

$$\Rightarrow \left(\frac{k}{2}\right)^2 = 0.05$$

$$\Rightarrow \frac{k}{2} = 0.2236.$$

Hence the rejection region is given by $\{Y < 0.2236\}$.