LECTURE - 29

Agenda:

1. Uniformly most powerful tests
2. Example

In the previous lecture, we learnt the Neyman-Pearson lemma. It is a very strong and interesting result which says that for testing a simple null hypothesis v.s. a simple alternative hypothesis there exists a test which has the highest power among all tests with level-α. Moreover, this test has rejection region of the form $\left\{ \frac{L(\theta_0)}{L(\theta_A)} \leq k \right\}$ where k is determined by the restriction that the level is α.

We now ask the following question. If we want to test

$$\underbrace{H_0 : \theta = \theta_0}_{\text{SIMPLE NULL}} \quad \text{v.s.} \quad \underbrace{H_A : \theta > \theta_0}_{\text{COMPOSITE ALTERNATIVE}},$$

is a result similar to the Neyman-Pearson lemma available for this situation? More precisely, is there a testing procedure which maximizes Power(θ_A) FOR EVERY $\theta_A > \theta_0$ among all testing procedures with level-α?

$$\boxed{\text{IN GENERAL, THE ANSWER IS NO.}}$$

However, for a large class of examples, we can use the Neyman-Pearson lemma to obtain such a procedure. Here is an example to illustrate these ideas.

<u>EXAMPLE:</u> Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from a Normal$(\mu, 1)$ population. Suppose we wish to test $H_0: \mu = \mu_0$ v.s. $H_A: \mu > \mu_0$.

Consider the simpler problem of testing

$$H_0: \mu = \mu_0 \quad \text{v.s.} \quad H_A': \mu = \mu_A,$$

where $\mu_A > \mu_0$ is arbitrarily fixed. The Neyman-Pearson lemma tells us that the most powerful level-α test has a rejection region given by

$$\left\{ \frac{L(\mu_0)}{L(\mu_A)} \le k \right\}.$$

Note that

$$L(\mu_0) = \prod_{i=1}^{n} f_{Y_i}(y_i \mid \mu_0)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\mu_0)^2}{2}}$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n}(y_i-\mu_0)^2/2}.$$

Similarily,

$$L(\mu_A) = \prod_{i=1}^{n} f_{Y_i}(y_i \mid \mu_A)$$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n}(y_i-\mu_A)^2/2}$$

Hence,

$$\frac{L(\mu_0)}{L(\mu_A)} \le k \iff \frac{\frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n}(y_i-\mu_0)^2/2}}{\frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n}(y_i-\mu_A)^2/2}} \le k$$

$$\iff e^{-\frac{1}{2}\left\{\sum_{i=1}^{n}(y_i-\mu_0)^2 - \sum_{i=1}^{n}(y_i-\mu_A)^2\right\}} \le k$$

$$\iff -\frac{1}{2}\left\{\sum_{i=1}^{n}(y_i-\mu_0)^2 - \sum_{i=1}^{n}(y_i-\mu_A)^2\right\} \le \log k$$

$$\iff \sum_{i=1}^{n}(y_i^2 - 2\mu_0 y_i + \mu_0)^2 - \sum_{i=1}^{n}(y_i^2 - 2\mu_0(y_i) \ge -2\log k$$

Hence,

$$\frac{L(\mu_0)}{L(\mu_A)} \le k \iff 2(\mu_A - \mu_0)\sum_{i=1}^{n} y_i + n\mu_0^2 - n\mu_A^2 \ge -2\log k$$

$$\iff 2n(\mu_A - \mu_0)\,\overline{y} \ge -2\log k - n\mu_0^2 + n\mu_A^2$$

$$\iff \overline{y} \ge \underbrace{\frac{-2\log k - n\mu_0^2 + n\mu_A^2}{2n(\mu_A - \mu_0)}}_{k'}$$

Hence, as expected, the rejection region looks like $\{\overline{y} > k'\}$, i.e., the null hypothesis will be rejected for large values of $\overline{y}$. Since the level is α,

$$P(H_0 \text{ is rejected} \mid H_0 \text{ is true}) = \alpha$$

$$\Rightarrow P(\overline{Y} \ge k' \mid \mu = \mu_0) = \alpha$$

$$\Rightarrow P\left(\sqrt{n}(\overline{Y} - \mu_0) \ge \sqrt{n}(k' - \mu_0) \mid \mu = \mu_0\right) = \alpha.$$

$$\Rightarrow P\left(\text{Normal}(0,1) \le \sqrt{n}(k' - \mu_0)\right) = 1 - \alpha.$$

($\because$ Note that $\overline{Y}$ is Normal$\left(\mu_0, \frac{1}{n}\right)$, and

hence $\sqrt{n}(\overline{Y} - \mu_0)$ is Normal$(0,1)$).

Hence, $\sqrt{n}(k' - \mu_0)$ HAS TO BE THE $(1-\alpha)^{th}$ QUANTILE OF A NORMAL$(0,1)$ RANDOM VARIABLE, i.e.,

$$\sqrt{n}(k' - \mu_0) = z_{1-\alpha}.$$

Hence,

$$k' = \mu_0 + \frac{z_{1-\alpha}}{\sqrt{n}}.$$

Hence, the most powerful level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu = \mu_A$ has a rejection region of the form

$$\left\{ \bar{Y} \geq \mu_0 + \frac{z_{1-\alpha}}{\sqrt{n}} \right\}$$

NOTE THAT THE REJECTION REGION IS COMPLETELY INDEPENDENT OF μ_A, I.E., FOR ANY $\mu_A > \mu_0$, WE OBTAIN THE SAME REJECTION REGION.

Returning to testing the hypothesis $H_0: \mu = \mu_0$ v.s. $H_A: \mu > \mu_0$, the level-α test with rejection region

$$\left\{ \bar{Y} \geq \mu_0 + \frac{z_{1-\alpha}}{\sqrt{n}} \right\}$$

has the highest power for every specific alternative $\mu_A > \mu_0$. SUCH A TEST IS CALLED THE UNIFORMLY MOST POWERFUL LEVEL-α TEST FOR TESTING $H_0: \mu = \mu_0$ V.S. $H_A: \mu > \mu_0$.

There is a wide class of examples, where the most powerful level-α tests are the same for every possible alternative $\theta_A > \theta_0$. In such a case, this test is known as the uniformly most powerful test for testing $H_0: \theta = \theta_0$ v.s. $H_A: \theta_A > \theta_0$.

REMARK 1: A similar conclusion holds for testing

$$H_0: \mu = \mu_0 \quad \text{v.s.} \quad H_A: \mu < \mu_0$$

in the case of the Normal $(\mu, 1)$ population. The level-α test with rejection region

$$\left\{ \bar{Y} \leq \mu_0 - \frac{z_{1-\alpha}}{\sqrt{n}} \right\}$$

is the uniformly most powerful level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu < \mu_0$.

REMARK 2: There does not exist any uniformly most powerful level-α test for testing $H_0: \mu = \mu_0$ v.s. $H_A: \mu \neq \mu_0$. In general, in most instances, there do not exist uniformly most powerful tests for two-sided alternative hypotheses of the form $H_A: \theta \neq \theta_0$.