LECTURE - 3

Agenda:

Examples to understand unbiasedness, MSE

EXAMPLE 1 (Exercise 8.10)

The number of breakdowns per week for a type of minicomputer is a random variable Y with a Poisson distribution with mean λ. A "random sample" $Y_1, Y_2, ---, Y_n$ of observations on the weekly number of breakdowns is available.

(a) Suggest an unbiased estimator for λ.

IMPORTANT: IN STANDARD STATISTICAL TERMINOLOGY, UNLESS OTHERWISE STATED, A "RANDOM SAMPLE $Y_1, Y_2, ---, Y_n$ FROM A POPULATION" MEANS THAT $Y_1, Y_2, ---, Y_n$ ARE INDEPENDENT AND HAVE THE SAME DISTRIBUTION.

In this case, it means that $Y_1, Y_2, ---, Y_n$ are independent and have a Poisson distribution with mean λ.

Note that

$$E[Y_i] = \lambda \quad \text{for each } i = 1, 2, ---, n.$$

Hence, the most obvious unbiased estimator of λ is

$$\hat{\lambda}_{avg} = \frac{1}{n} \sum_{i=1}^{n} Y_i = \overline{Y}.$$

In statistics, the average of random variables $Y_1, Y_2, ---, Y_n$ is often represented by $\overline{Y}$.

Note that

$$E[\hat{\lambda}_{avg}] = E[\overline{Y}] = \frac{1}{n} \sum_{i=1}^{n} E[Y_i] = \frac{1}{n} \sum_{i=1}^{n} \lambda = \lambda.$$

Hence $\hat{\lambda}_{avg}$ is an unbiased estimate of λ.

However, one may just $\hat{\lambda}_1 = Y_1$ as an estimator for λ.
Note that

$$E[\hat{\lambda}_1] = E[Y_1] = \lambda.$$

Hence $\hat{\lambda}_1$ is also an unbiased estimator for λ. So why would we not work with $\hat{\lambda}_1$. The reason is simple. Choose the estimator with the smaller MSE.

Since $\hat{\lambda}_{avg}$ and $\hat{\lambda}_1$ are both unbiased,

$$MSE(\hat{\lambda}_{avg}) = Var(\hat{\lambda}_{avg}) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} Y_i\right) = \frac{\lambda}{n},$$

$$MSE(\hat{\lambda}_1) = Var(\hat{\lambda}_1) = V(Y_1) = \lambda.$$

Hence, $\hat{\lambda}_{avg}$ always has a lower MSE than $\hat{\lambda}_1$.

(b) The weekly cost of repairing these breakdowns is

$$C = 3Y + Y^2.$$

Show that $E[C] = 4\lambda + \lambda^2$.

Note that

$$E[C] = E[3Y + Y^2] = 3E[Y] + E[Y^2]$$

$$= 3\lambda + V(Y) + (E[Y])^2$$

$$= 3\lambda + \lambda + \lambda^2$$

$$= 4\lambda + \lambda^2.$$

(c) Find a function of $Y_1, Y_2, \ldots, Y_n$ that is an unbiased estimator of C.

Note that by our computation in Part (A),

$$E[3Y_i + Y_i^2] = 4\lambda + \lambda^2 = E[C] \text{ for each } i = 1, 2, \dots, n.$$

Hence,

$$\frac{1}{n} \sum_{i=1}^{n} E[3Y_i + Y_i^2) = E[C)$$

$$\Rightarrow E\left[\frac{1}{n} \sum_{i=1}^{n} (3Y_i + Y_i^2)\right] = E[C].$$

Hence $\widehat{E[C]} = \frac{1}{n} \sum_{i=1}^{n} (3Y_i + Y_i^2)$ is an unbiased estimate of $E[C]$.