LECTURE - 5

Agenda:

1. Some clarifications about Homework 1
2. Unbiased estimators in some standard problems

SOME CLARIFICATIONS ABOUT HOMEWORK 1

We will study the technique to find the density of the minimum or maximum of independent random variables which have a common distribution.

RESULT: Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with a common distribution function F and density function f. Then, the density function of

$Y_{(1)} \triangleq \min(Y_1, Y_2, \ldots, Y_n)$ is given by

$$f_{Y_{(1)}}(y) = n\big(1 - F(y)\big)^{n-1} f(y),$$

and the density function of

$Y_{(n)} \triangleq \max(Y_1, Y_2, \ldots, Y_n)$ is given by

$$f_{Y_{(n)}}(y) = n\big(F(y)\big)^{n-1} f(y).$$

Let us use this result to solve the following problem.

<u>Question:</u> If Y_1, Y_2, Y_3 are independent exponential random variables with mean θ, find the bias and MSE of the estimator $\hat{\theta} = \min(Y_1, Y_2, Y_3)$.

<u>Solution:</u> From the result on the previous page, the density of $\hat{\theta}$ is given by

$$f_{\hat{\theta}}(y) = 3(1 - F(y))^{3-1} f(y).$$

Note that $F(y) = \begin{cases} 1 - e^{-y/\theta} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$,

and $f(y) = \begin{cases} \dfrac{e^{-y/\theta}}{\theta} & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$

Hence,

$$f_{\hat{\theta}}(y) = \begin{cases} \dfrac{3}{\theta} e^{-\frac{3y}{\theta}} & \text{if } y \geq 0, \\ 0 & \text{otherwise} \end{cases}.$$

It follows that $\hat{\theta}$ has an exponential distribution with mean $\frac{\theta}{3}$. Hence, the bias of $\hat{\theta}$ is given by

$$B(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{\theta}{3} - \theta = -\frac{2\theta}{3},$$

and the MSE of $\hat{\theta}$ is given by

$$MSE(\hat{\theta}) = V(\hat{\theta}) + (B(\hat{\theta}))^2$$

$$= \left(\frac{\theta}{3}\right)^2 + \left(-\frac{2\theta}{3}\right)^2$$

($\because$ Variance of an Exponential random variable is the square of the mean.)

$$= \frac{5\theta^2}{9}.$$

UNBIASED ESTIMATOR OF DIFFERENCE OF MEANS IN TWO POPULATIONS

Suppose we have two populations.

Population 1 has mean μ_1 and variance σ_1^2.
Population 2 has mean μ_2 and variance σ_2^2.

Let $Y_1, Y_2, \ldots, Y_{n_1}$ be a random sample from Population 1, and $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_{n_2}$ be a random sample from Population 2.

Problem: Provide a statistical estimator for $\mu_1 - \mu_2$. Also provide the bias and standard error of your estimator.

The most obvious estimator for μ_1 is $\underbrace{\overline{Y}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} Y_i}_{\text{sample mean for first population}}$

The most obvious estimator for μ_2 is $\underbrace{\bar{Y}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} \tilde{Y}_i}_{\text{Sample mean for second population.}}$

Hence, the most obvious estimatok for $\mu_1 - \mu_2$ is $\bar{Y}_1 - \bar{Y}_2$.

Bias: $E[\bar{Y}_1 - \bar{Y}_2] = E[\bar{Y}_1] - E[\bar{Y}_2] = \mu_1 - \mu_2$.

Hence $\bar{Y}_1 - \bar{Y}_2$ is unbiased for $\mu_1 - \mu_2$.

Standard error: $MSE(\bar{Y}_1 - \bar{Y}_2) = Var(\bar{Y}_1 - \bar{Y}_2)$ (By unbiasedness)

$$= Var(\bar{Y}_1) + Var(\bar{Y}_2)$$

$$= Var\left(\frac{1}{n_1} \sum_{i=1}^{n_1} Y_i\right) + Var\left(\frac{1}{n_2} \sum_{i=1}^{n_2} \tilde{Y}_i\right)$$

$$= \frac{1}{n_1^2} \sum_{i=1}^{n_1} Var(Y_i) + \frac{1}{n_2^2} \sum_{i=1}^{n_2} Var(\tilde{Y}_i)$$

$$= \frac{1}{n_1^2} \sum_{i=1}^{n_1} \sigma_1^2 + \frac{1}{n_2^2} \sum_{i=1}^{n_2} \sigma_2^2$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}.$$

Hence, $SE(\bar{Y}_1 - \bar{Y}_2) = \sqrt{MSE(\bar{Y}_1 - \bar{Y}_2)}$

$$= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}.$$