LECTURE - 6

Agenda:

1. Random sample from normal population
2. The χ^2-distribution
3. The t-distribution

$Y_1, Y_2, ---, Y_n$

Suppose we have a random sample from a population which is Normal with mean μ and variance σ^2.

We know that an unbiased estimate of the population mean μ is given by

$$\hat{\mu} = \bar{Y} \triangleq \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{(the sample mean).}$$

The first question that we strive to answer is: If the random sample has a common normal distribution, what is the distribution of $\hat{\mu}$??

RESULT (from 432a): If $Y_1, Y_2, ---, Y_n$ are independent random variables which have a common Normal(μ, σ^2) distribution, then

$$\boxed{\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \text{ has a Normal}\left(\mu, \frac{\sigma^2}{n}\right) \text{ distribution}}$$

We had derived (without even assuming that the common distribution is normal) that

$$MSE(\bar{Y}) = \frac{\sigma^2}{n}.$$

The result on the previous page verifies this computation, as $\bar{Y} \sim \text{Normal}(\mu, \frac{\sigma^2}{n}) \Rightarrow MSE(\bar{Y}) = V(\bar{Y}) = \frac{\sigma^2}{n}$.

We had also derived that an unbiased estimator of the population variance σ^2 is given by the corrected sample variance

$$S^2 \triangleq \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2.$$

<u>Question</u>: What is $MSE(S^2)$??

It is hard to answer this question in general, but if we assume that the common distribution is normal, we have answers to this question.

DEFINITION: If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then the distribution of the random variable $\sum_{i=} Z_i^2$ is defined to have a **χ^2-distribution with n degrees of freedom**.

FACT: The χ^2-distribution with n degrees of freedom is infact the same as the Gamma distribution with parameters $\alpha = \frac{n}{2}$ and $\beta = 2$.

RESULT: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population which is normal with mean μ and variance σ^2. Then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

has a χ^2 distribution with $(n-1)$ degrees of freedom. Also, $\bar{Y}$ and S^2 are independent random variables.

Note that the expected value of a Gamma(α, β) random variable is given by $\alpha\beta$ and the variance is given by $\alpha\beta^2$. Hence,

$$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{(n-1)}{2} \times 2 = (n-1)$$

$$\Rightarrow E[S^2] = \sigma^2.$$

This verifies our derivation of the unbiasedness of S^2 (without even assuming that the common distribution is normal). Also,

$$MSE(S^2) = V(S^2) = \frac{\sigma^4}{(n-1)^2} V\left(\frac{(n-1)S^2}{\sigma^2}\right).$$

Hence,

$$MSE(S^2) = \frac{\sigma^4}{(n-1)^2} \cdot \frac{(n-1)}{2} \times 2^2$$

$$= \frac{2\sigma^4}{(n-1)}.$$

In this context, another important distribution that comes up is the t-distribution.

DEFINITION: Let Z be a standard normal random variable and let W be a χ^2-distributed random variable with ν degrees of freedom. Then, if Z and W are independent,

$$T = \frac{Z}{\sqrt{W/\nu}}$$

is defined to have a t-distribution with ν degrees of freedom.

RESULT: If $Y_1, Y_2, \ldots, Y_n$ is a random sample from a population which is normal with mean μ and variance σ^2, then

$$\frac{\sqrt{n}(\bar{Y}-\mu)}{S}$$ has a t-distribution with $(n-1)$ degrees of freedom.