LECTURE - 7

Agenda:

1. Another criterion for evaluating quality of estimators
2. Examples

Let $\hat{\theta}$ be a statistical estimator of a parameter θ. Recall that there are two ways through which we evaluate the quality of a statistical estimator

1. UNBIASEDNESS ($E[\hat{\theta}] = \theta$).
2. $MSE(\hat{\theta}) = E[(\hat{\theta}-\theta)^2]$ should be low.

Another criterion that people often use is "ERROR PROBABILITIES". For a fixed $\varepsilon > 0$, they ask the question: What is the probability that $\hat{\theta}$ will be within ε distance of θ? Mathematically, they want us to evaluate

$$P(|\hat{\theta} - \theta| < \varepsilon).$$

The higher this probability, the better the quality of the estimator $\hat{\theta}$.

Example: A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1.0$ ounce. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottled with the same machine setting), and the ounces of fill are measured for each. Find the probability that the sample mean will be within 0.3 ounce of the true population mean μ for the chosen machine setting.

Solution: Let $Y_1, Y_2, ---, Y_9$ denote the ounces of fill to be observed in the 9 samples respectively. Then we know that $Y_1, Y_2 ---, Y_9$ are independent and have a common Normal$(\mu, 1)$ distribution. We want to evaluate the following probability.

$$P(|\bar{Y} - \mu| < 0.3).$$

Note that $\bar{Y}$ is Normal$\left(\mu, \frac{\sigma^2}{n}\right)$ = Normal$\left(\mu, \frac{1}{9}\right)$ from the result in Lecture 6. Remember from 4321 that whenever we have a computation related to a normal random variable, the standard way to proceed is to convert to standard normal.

RESULT (4321) If X is Normal(μ, σ^2), then $Z = \frac{X-\mu}{\sigma}$ is Normal$(0,1)$.

Hence, in our case $Z = \dfrac{\bar{Y} - \mu}{\sqrt{1/9}}$ is Normal$(0,1)$.

Hence,
$$P(|\bar{Y} - \mu| < 0.3) = P\left(\left|\frac{\bar{Y} - \mu}{\sqrt{1/9}}\right| < \frac{0.3}{\sqrt{1/9}}\right)$$
$$= P(|Z| < 0.9).$$

Note that we denote the distribution function of a standard normal random variable by Φ, i.e.,

$$\Phi(z) = P(Z \leq z) .$$

Hence,
$$P(|Z| < 0.9) = P(-0.9 < Z \leq 0.9)$$
$$= P(Z \leq 0.9) - P(Z \leq -0.9)$$
$$= \Phi(0.9) - \Phi(-0.9).$$
$$= 0.6318 .$$

The Φ-function can be evaluated from standard tables or statistical software. For this course, you are only expected to know the following facts.

$$\Phi(0) = \frac{1}{2} .$$

$$\Phi(1.68) - \Phi(-1.68) = 90\% .$$

$$\Phi(1.96) - \Phi(-1.96) = 95\% .$$

<u>RESULT</u> : If $Y_1, Y_2, \ldots, Y_n$ are independent random variables with a common Normal(μ, σ^2) distribution, then

$$P(|\bar{Y} - \mu| < \varepsilon) = \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{\varepsilon\sqrt{n}}{\sigma}\right) .$$

Note that to use this formula, <u>we assume σ is known.</u>

Example: In the earlier example, find the number of observations or samples n such that

$$P(|\bar{Y}-\mu| < 0.3) = 0.95.$$

From the result,

$$P(|\bar{Y}-\mu| < 0.3) = \Phi(0.3\sqrt{n}) - \Phi(-0.3\sqrt{n})$$

We want n such that

$$\Phi(0.3\sqrt{n}) - \Phi(-0.3\sqrt{n}) = 0.95$$

But we already know that

$$\Phi(1.96) - \Phi(-1.96) = 0.95$$

Hence,

$$1.96 = 0.3\sqrt{n}$$

$$\Rightarrow \quad n = \left(\frac{1.96}{3}\right)^2 = 42.68$$

Hence $n = 43$ samples are requaired for $P(|\bar{Y}-\mu| < 0.3) = 0.95$

Note that $P(|\bar{Y}-\mu| < 0.3) = \Phi(0.3\sqrt{n}) - \Phi(-0.3\sqrt{n})$ is an increasing function of n.