LECTURE- 9

Agenda:

1. Confidence interval in normal population with UNKNOWN σ.
2. Upper confidence limit.
3. Lower confidence limit.
4. Examples

CONFIDENCE INTERVAL IN NORMAL POPULATION WITH UNKNOWN σ.

In the last lecture we considered the scenario, where we have a random sample $Y_1, Y_2, \ldots, Y_n$ from a population which is Normal (μ, σ^2). We are interested in a 95% confidence interval for μ. If σ is known, we used $\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}$ as the pivotal quantity to obtain the

confidence interval $\left[\bar{Y} - \frac{1.96\sigma}{\sqrt{n}}, \bar{Y} + \frac{1.96\sigma}{\sqrt{n}}\right]$.

WHAT IF σ is UNKNOWN? Let $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$.

1. Recall that $\frac{\sqrt{n}(\bar{Y}-\mu)}{S}$ has a t-distribution with $n-1$ degrees of freedom. Since this distribution is independent of μ, $g(\text{Data}, \mu) = \frac{\sqrt{n}(\bar{Y}-\mu)}{S}$ is a pivotal quantity.

(2) We want a and b such that

$$P\left(a \le \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \le b \right) = 95\%.$$

FACT: Let $t_{n-1,\,1-\alpha}$ denote the $(1-\alpha)^{th}$ quantile of the t-distribution with n-1 degrees of freedom, for every $0 \le \alpha \le 1$.

By the symmetry of the t-distribution around 0, it is known that for every $0 \le \alpha \le 1$,

$$P\left(-t_{n-1,\,1-\frac{\alpha}{2}} \le \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \le t_{n-1,\,1-\frac{\alpha}{2}} \right) = 1-\alpha.$$

We want $1-\alpha = 95\%$ in this particular problem.
Hence $1-\frac{\alpha}{2} = 97.5\%$. We get that

$$P\left(-t_{n-1,\,97.5\%} \le \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \le t_{n-1,\,97.5\%} \right) = 95\%.$$

(3) $$-t_{n-1,\,97.5\%} \le \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \le t_{n-1,\,97.5\%}$$

$$\Longleftrightarrow \frac{-t_{n-1,\,97.5\%}\,S}{\sqrt{n}} \le \bar{Y}-\mu \le \frac{t_{n-1,\,97.5\%}\,S}{\sqrt{n}}$$

$$\Longleftrightarrow \bar{Y} - \frac{t_{n-1,\,97.5\%}\,S}{\sqrt{n}} \le \mu \le \bar{Y} + \frac{t_{n-1,\,97.5\%}\,S}{\sqrt{n}}$$

Hence, a 95% confidence interval for μ is given by $\left[\bar{Y} - \frac{t_{n-1, 97.5\%} S}{\sqrt{n}} , \bar{Y} + \frac{t_{n-1, 97.5\%} S}{\sqrt{n}} \right]$

UPPER CONFIDENCE LIMIT

<u>DEFINITION:</u> We say that $\hat{\theta}_{upper}$ is an upper confidence limit for an unknown parameter θ, with confidence level $1-\alpha$, if

$$P(\theta \leq \hat{\theta}_{upper}) = 1-\alpha.$$

Method to find upper confidence limit with confidence level $1-\alpha$

(1) Get a pivotal quantity $g(\text{Data}, \theta)$ WHICH IS AN INCREASING FUNCTION OF θ.

(2) Find the $(1-\alpha)^{th}$ quantile of $g(\text{Data}, \theta)$, i.e., find b such that

$$P(g(\text{Data}, \theta) \leq b) = 1-\alpha.$$

(3) Reduce $g(\text{Data}, \theta) \leq b$ to the form $\theta \leq \hat{\theta}_{upper}$.

Let us try this method to find an upper confidence limit for μ in the normal population problem with KNOWN σ.

(1) The pivotal quantity we generally work with is $\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}$. However since we want the pivotal quantity to be an increasing function of μ, we just

choose the negative, i.e., $\frac{\sqrt{n}(\mu-\bar{Y})}{\sigma}$ as our pivotal quantity. Suppose the confidence level that we desire is $1-\alpha$.

(2) We want b such that

$$P\left(\frac{\sqrt{n}(\mu-\bar{Y})}{\sigma} \le b\right) = 1-\alpha$$

$$\Leftrightarrow P\left(-\frac{\sqrt{n}(\mu-\bar{Y})}{\sigma} \ge -b\right) = 1-\alpha$$

$$\Leftrightarrow P\left(\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \ge -b\right) = 1-\alpha$$

$$\Leftrightarrow P\left(\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \le -b\right) = \alpha.$$

Hence, $-b$ is the α^{th} quantile of $Z = \frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}$, which we know has a Normal$(0,1)$ distribution.

The standard notation for the α^{th} quantile of the Normal$(0,1)$ distribution is z_α for every $0 \le \alpha \le 1$.

Hence,
$$-b = z_\alpha \Rightarrow b = -z_\alpha \boxed{= z_{1-\alpha}}$$

FACT: By the symmetry of the Normal$(0,1)$ distribution.

(3) $\frac{\sqrt{n}(\mu-\bar{Y})}{\sigma} \le b \Leftrightarrow \mu \le \bar{Y} + \frac{b\sigma}{\sqrt{n}} = \bar{Y} + \frac{z_{1-\alpha}\sigma}{\sqrt{n}}$

Hence, a $(1-\alpha)$ upper confidence limit for μ is given by $\hat{\mu}_{upper} = \bar{Y} + \frac{z_{1-\alpha}\,\sigma}{\sqrt{n}}$.

LOWER CONFIDENCE LIMIT

DEFINITION: We say that $\hat{\theta}_{lower}$ is a lower confidence limit for an unknown parameter θ with confidence level $1-\alpha$, if

$$P(\hat{\theta}_{lower} \leq \theta) = 1-\alpha.$$

Method to find lower confidence limit with confidence level $1-\alpha$

(1) Get a pivotal quantity $g(\text{Data}, \theta)$ WHICH IS AN INCREASING FUNCTION OF θ.

(2) Find b such that

$$P(b \leq g(\text{Data}, \theta)) = 1-\alpha.$$

(3) Reduce $b \leq g(\text{Data}, \theta)$ to the form $\hat{\theta}_{lower} \leq \theta$.

Let us try this method to find a lower confidence limit for μ in the normal population problem with KNOWN σ.

(1) Again, we use the pivotal quantity $\frac{\sqrt{n}(\mu-\bar{Y})}{\sigma}$ to make sure it is an increasing function of μ.

(2) We want to find b such that

$$P\left(b \leq \frac{\sqrt{n}(\mu - \bar{Y})}{\sigma} \right) = 1 - \alpha$$

$$\Leftrightarrow P\left(\frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \leq -b \right) = 1 - \alpha \qquad (\text{Normal}(0,1))$$

$\Leftrightarrow -b = z_{1-\alpha}$ (the $(1-\alpha)^{th}$ quantile of the Normal$(0,1)$ distribution)

$\Leftrightarrow b = -z_{1-\alpha}$

(3) $$b \leq \frac{\sqrt{n}(\mu - \bar{Y})}{\sigma} \Leftrightarrow \bar{Y} + \frac{b\sigma}{\sqrt{n}} \leq \mu \Leftrightarrow \bar{Y} - \frac{z_{1-\alpha}\sigma}{\sqrt{n}} \leq \mu.$$

Hence, a $(1-\alpha)$ lower confidence limit for μ is given by

$$\hat{\mu}_{lower} = \bar{Y} - \frac{z_{1-\alpha}\sigma}{\sqrt{n}}.$$