SOLUTIONS TO SAMPLE EXAM

Problem 1: (a) The standard unbiased estimate for the population mean μ is the sample mean $\bar{Y}$.

(b) The mean squared error of $\bar{Y}$ is $\frac{\sigma^2}{n}$.

Since $\sigma^2 = 1600$ and $n = 100$,

$$MSE(\bar{Y}) = \frac{1600}{100} = 16.$$

(c) Note that $Z = \frac{\sqrt{n}\,(\bar{Y} - \mu)}{\sigma}$ is Normal(0,1).

Hence,

$$P(|\bar{Y} - \mu| < 0.49) = P\left(\left|\frac{\sqrt{n}\,(\bar{Y}-\mu)}{\sigma}\right| < \frac{\sqrt{n}\,0.49}{\sigma}\right)$$

$$= P(|Z| < 1.96)$$

$(\because \sigma^2 = 1600,\ n = 100)$

$$= \Phi(1.96) - \Phi(-1.96)$$

$$= 95\%.$$

Problem 2: $\boxed{(a)}$ The mean squared error of S^2 is given by

$$MSE(S^2) = \frac{2\sigma^4}{n-1}.$$

$\boxed{(b)}$ The distribution of $\frac{(n-1)S^2}{\sigma^2}$ is a χ^2-distribution with $n-1$ degrees of freedom.

$\boxed{(c)}$ The χ^2-distribution with $(n-1)$ degrees of freedom is the same as a Gamma distribution with $\alpha = \frac{n-1}{2}$ and $\beta = 2$.

Hence,

$$E\left[\left(\frac{(n-1)S^2}{\sigma^2}\right)^2\right] = \alpha(\alpha+1)\beta^2$$

$$= \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)2^2$$

$$= (n-1)(n+1)$$

$$\Rightarrow E\left[\frac{(n-1)^2 S^4}{\sigma^4}\right] = (n-1)(n+1)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} E[S^4] = (n-1)(n+1)$$

$$\Rightarrow E[S^4] = \frac{(n+1)}{(n-1)}\sigma^4$$

ALTERNATIVE APPROACH:

Note that

$$MSE(s^2) = E[(s^2 - \sigma^2)^2]$$

$$= E[(s^2 - (E(s^2)))^2]$$

($\because$ S^2 is unbiased for σ^2)

$$= E[s^4] - (E[s^2])^2$$

($\because$ By the properties of variance)

$$= E[s^4] - \sigma^4.$$

Hence,

$$\frac{2\sigma^4}{n-1} = E[s^4] - \sigma^4.$$

$$\Rightarrow E[s^4] = \frac{(n+1)}{(n-1)} \sigma^4.$$

Problem 3: $\boxed{(a)}$ Step 1 of the pivotal quantity method has already been provided to us, i.e., we know that $W = \frac{2}{\theta} \sum_{i=1}^{100} Y_i$ is a pivotal quantity, and has a χ^2-distribution with 200 degrees of freedom.

Step 2: Find a and b such that

$$P(a \le W \le b) = 95\%.$$

Again, it has been given that

$$P(1 \le W \le 235) = 95\%.$$

Hence $a = 1$ and $b = 235$.

Step 3: Express $a \le W \le b$ as $\hat{\theta}_L \le \theta \le \hat{\theta}_U$.

Note that

$$\{1 \le W \le 235\}$$

$$\Leftrightarrow \left\{ 1 \le \frac{2\sum_{i=1}^{100} Y_i}{\theta} \le 235 \right\}$$

$$\Leftrightarrow \left\{ \frac{1}{235} \le \frac{\theta}{2\sum_{i=1}^{100} Y_i} \le 1 \right\}$$

$$\Leftrightarrow \left\{ \frac{2\sum_{i=1}^{100} Y_i}{235} \le \theta \le 2\sum_{i=1}^{100} Y_i \right\}.$$

Hence, a 95% confidence interval for θ is given by

$$\left[\frac{2\sum_{i=1}^{100} Y_i}{235}, \; 2\sum_{i=1}^{100} Y_i \right].$$

$\boxed{(b)}$ If $\bar{Y} = \frac{1}{100}\sum_{i=1}^{100} Y_i = 80$

$\Rightarrow \sum_{i=1}^{\cdot} Y_i = 8000$

Hence the confidence interval for θ is given by

$$\left[\frac{2\times 8000}{235}, 2\times 8000\right] = \left[68.085, 16000\right].$$

Problem 4: (a) Step 1 of the pivotal quantity method has already been provided to us, i.e., we know that $W = \frac{2}{\beta}\sum_{i=1}^{100} Y_i$ is a pivotal quantity, and has a χ^2-distribution with 400 degrees of freedom.

Step 2: Find a and b such that

$$P(a \leq W \leq b) = 95\%.$$

Again, it has been given that

$$P(1 \leq W \leq 448) = 95\%.$$

Hence $a = 1$ and $b = 448$.

Step 3: Express $a \leq W \leq b$ as $\hat{\theta}_L \leq \theta \leq \hat{\theta}_U$.

Note that

$$\{1 \le W \le 448\}$$

$$\Leftrightarrow \left\{1 \le \frac{2}{\beta}\sum_{i=1}^{100} Y_i \le 448\right\}$$

$$\Leftrightarrow \left\{\frac{1}{448} \le \frac{\beta}{2\sum_{i=1}^{100} Y_i} \le 1\right\}$$

$$\Leftrightarrow \left\{\frac{\sum_{i=1}^{100} Y_i}{224} \le \beta \le 2\sum_{i=1}^{100} Y_i\right\}.$$

Hence, a 95% confidence interval for θ is given by

$$\left[\frac{\sum_{i=1}^{100} Y_i}{224}, 2\sum_{i=1}^{100} Y_i\right].$$

$\boxed{(b)}$ If $\bar{Y} = \frac{1}{100}\sum_{i=1}^{100} Y_i = 200$, then the 95% confidence interval for θ is given by

$$\left[\frac{100 \times 200}{224}, 200 \times 200\right] = [89.286, 20000].$$