# SOLUTIONS TO SAMPLE EXAM

**Problem 1:** $\boxed{(a)}$ We know that if $n$ is large enough, then the distribution of $\frac{\sqrt{n}(\overline{Y}-\mu)}{\sigma}$ is approximately Normal$(0,1)$.

**Step 1:** $\frac{\sqrt{100}(\overline{Y}-\mu)}{\sqrt{1600}} = \frac{\overline{Y}-\mu}{4}$ has an approximate Normal$(0,1)$ distribution, and hence can be used as a pivotal quantity.

**Step 2:** We need to find $a$ and $b$ such that

$$P\left(a \le \frac{\overline{Y}-\mu}{4} \le b\right) = 0.95.$$

Since $\frac{\overline{Y}-\mu}{4}$ has a Normal$(0,1)$ distribution, we know that $a=-1.96$, $b=1.96$.

**Step 3:** $$-1.96 \le \frac{\overline{Y}-\mu}{4} \le 1.96$$

$$\Leftrightarrow \overline{Y}-7.84 \le \mu \le \overline{Y}+7.84$$

Hence, an approximate large sample confidence interval with confidence level 95% for $\mu$ is given by

$$\left[\bar{Y} - 7.84,\ \bar{Y} + 7.84\right].$$

**(b)** If $\bar{Y} = 30$, then an approximate large sample confidence interval for $\mu$ with confidence level 95% is given by

$$[22.16,\ 37.84].$$

**Problem 2:** **(a)** 
$$E[\hat{\theta}_1] = E\left[\frac{Y_1 + Y_2}{2}\right]$$
$$= \frac{E[Y_1] + E[Y_2]}{2}$$
$$= \frac{\theta + \theta}{2}$$
$$= \theta.$$

$$E[\hat{\theta}_2] = E[\bar{Y}] = E\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \frac{\theta + \theta + \theta}{3} = \theta.$$

Hence $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of $\theta$.

$\boxed{(b)}$ $MSE(\hat{\theta}_1) = V(\hat{\theta}_1)$ $(\because \hat{\theta}_1$ is unbiased$)$

$$= V\left(\frac{Y_1+Y_2}{2}\right)$$

$$= \frac{V(Y_1)+V(Y_2)}{4} \quad (\because \text{By independence})$$

$$= \frac{\theta^2+\theta^2}{4}$$

$$= \frac{\theta^2}{2}.$$

$MSE(\hat{\theta}_2) = V(\hat{\theta}_2)$ $(\because \hat{\theta}_2$ is unbiased$)$

$$= V\left(\frac{Y_1+Y_2+Y_3}{3}\right)$$

$$= \frac{V(Y_1)+V(Y_2)+V(Y_3)}{9} \quad (\because \text{By independence})$$

$$= \frac{\theta^2+\theta^2+\theta^2}{9}$$

$$= \frac{\theta^2}{3}.$$

Hence $\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \dfrac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)} = \dfrac{\theta^2/3}{\theta^2/2} = \dfrac{2}{3}.$

Problem 3: $\boxed{(a)}$

$$E[Y_1] = \int_{-\infty}^{\infty} y f(y)\, dy$$

$$= \int_0^1 y \theta y^{\theta-1}\, dy$$

$$= \theta \int_0^1 y^{\theta}\, dy$$

$$= \theta \left[ \frac{y^{\theta+1}}{\theta+1} \right]_0^1$$

$$= \frac{\theta}{\theta+1}.$$

$\boxed{(b)}$ The only way that we have of showing consistency is to show that $MSE(\bar{Y})$ converges to 0 as $n$ converges to infinity.

Note that $E[\bar{Y}] = \frac{1}{n} \sum_{i=1}^{n} E[Y_i] = \frac{\theta}{\theta+1}$.

Hence $Bias(\bar{Y}) = 0$, i.e., $\bar{Y}$ is an unbiased estimator of $\frac{\theta}{\theta+1}$. This implies

$$MSE(\bar{Y}) = V(\bar{Y}) = \frac{V(Y_1)}{n}$$

Now, $V(Y_1) = E[Y_1^2] - (E[Y_1])^2 = \int_{-\infty}^{\infty} y^2 f(y)\, dy - \left(\frac{\theta}{\theta+1}\right)^2$

$$\Rightarrow V(Y_1) = \theta \int_0^1 y^{\theta+1}\, dy - \left(\frac{\theta}{\theta+1}\right)^2$$

$$= \frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2 .$$

Hence,

$$MSE(\bar{Y}) = \frac{1}{n}\left(\frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2\right) \to 0 \text{ as } n \to \infty .$$

**Problem 4:** $\boxed{(a)}$ The only way that we have of showing sufficiency is using the factorization criterion

$$L(y_1, y_2, \ldots, y_n \mid \theta) = f_{Y_1, Y_2, \ldots, Y_n}(y_1, y_2, \ldots, y_n)$$

$$= \prod_{i=1}^{n} f_{Y_i}(y_i) \quad (\because \text{By independence})$$

$$= \prod_{i=1}^{n} \frac{1}{\theta}\, 1_{\{0 \le y_i \le \theta\}}$$

$$= \frac{1}{\theta^n}\, 1_{\{0 \le y_i \le \theta \text{ for every } 1 \le i \le n\}}$$

$$= \frac{1}{\theta^n}\, 1_{\{\max(y_1, y_2, \ldots, y_n) \le \theta,\ \min(y_1, y_2, \ldots, y_n) \ge 0\}}$$

$$= \underbrace{\frac{1}{\theta^n} \, 1_{\{\max(y_1, y_2, \dots, y_n) \le \theta\}}}_{g(y_{(n)},\, \theta)} \; \underbrace{1_{\{\min(y_1, y_2, \dots, y_n) \ge 0\}}}_{h(y_1, y_2, \dots, y_n)}$$

Hence, by the factorization criterion, $Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$ is a sufficient estimator for $\theta$.

$\boxed{(b)}$ We know that $Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$ is the minimal sufficient estimator for $\theta$. We would like to find a function of $Y_{(n)}$, which is unbiased for $\theta$. Note that

$$E\left[Y_{(n)}\right] = \left(\frac{n}{n+1}\right)\theta .$$

Hence,
$$E\left[\left(\frac{n+1}{n}\right) Y_{(n)}\right] = \theta .$$

Hence, $\left(\frac{n+1}{n}\right) Y_{(n)}$ is the MVUE estimator for $\theta$.