Solutions to Homework 2

2.41

There are $\binom{10}{4}$ ways to choose 4 engineers out of 10, a nd there are $4!$ ways to assign 4 different positions to 4 engineers chosen, thus

There are $\binom{10}{4} \cdot 4! = 5040$ ways can the director fill the positions.

2.50

For each of the 10 questions, sh e has two choices, tru e and false. Thus, there are 2^{10} ways for the student to finish the test . Because she is guessing, each way has the same probability.

If she gets exactly i correct, there are $\binom{10}{i}$ ways to specify i correct in 10 questions.

To pass the test, the student need exactly 7 or 8 or 9 or 10 correct, there are

$\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}$ ways to pass the test.

Therefore, the probability of passing the test is

$$\frac{\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}}{2^{10}} = \frac{11}{64}$$

2.62

There are $P_4^4 = 4!$ ways to arrange 4 m en with order, and there are $4!$ ways to arrange 4 women with order, thus there are $4! \times 4! = 576$ ways for the waiter to seat four couples. If four couples are seated across from one another, after we specify the m en's seats, women's seats are fixe d, thus th ere are $4! = 24$ possible outcom es that coupl es ar e all seated across from each other, the probability is

$$\frac{24}{576} = \frac{1}{24}$$

2.68

First we need to know the total number of pa tterns in a system of 8 particles and 10 phase cells. Cells are distingu ishable, w e can denote them by cellone, celltw o, …, cellten.

Each distribution of particles am ong phase cells can b e seen as on e outcom e of selecting 8 cells out of 10 cells with repl acement and without order . If one cell is

selected i times, we assign i particles to this cell.
For example, one outcome of selecting 8 cells out of 10 cells is: 3 cellone, 2 celltwo, 3 cellten. Then we assign 3 particles to cellone, 2 particles to celltwo, 3 particles to cellten, this is a distribution of 8 particles among 10 phase cells.

Therefore, there are $\binom{10+8-1}{8} = \binom{17}{8}$ distributions of 8 particles among 10 phase cells.

If all particles are in the same phase cell, there are 10 ways to specify the cell that contains all particles. The probability of interest is

$$\frac{10}{\binom{17}{8}} = \frac{1}{2431}$$

(See page 4 for problem 2.77, 2.80)

2.84

a. to select 6 senators out of 100 to form the committee, there are $\binom{100}{6}$ ways.

b. at most one senator from each state can serve on the committee, thus 6 committee members are from 6 different states. There are $\binom{50}{6}$ ways to specify the 6 states. And for each of the 6 states, there are 2 ways to select the senator to form the committee, thus, there are $\binom{50}{6} \times 2^6$ ways to form the committee.

2.88

Each passenger could get off on each of 8 floors, thus there are $8^5 = 32768$ possible outcomes in total.

If no two passengers get off on the same floor, there are $\binom{8}{5}$ ways to specify the 5 floors that have one passenger gets off, and there are $5!$ ways to arrange 5 passengers to get off on 5 floors with order, thus there are $\binom{8}{5} \cdot 5! = 6720$ ways to have no two passengers get off on the same floor.

Therefore, the probability of interest is

$$\frac{6720}{32768} = \frac{105}{512}$$

3.7

For a given driver, Let A denote the event that he/she is involved in fatal crashes, B denote the event that he/she is between 20 and 29 years of age, C denote the event that he/she had a blood alcohol level of at least 0.01. Then we have

$$P(B \mid A) = 0.28, P(C \mid AB) = 0.39$$

We need to compute $P(BC \mid A)$

$$P(BC \mid A) = \frac{P(ABC)}{P(A)} = \frac{P(ABC)}{P(AB)} \cdot \frac{P(AB)}{P(A)} = P(C \mid AB) \cdot P(B \mid A) = 0.39 \cdot 0.28 = 0.1092$$

2.77

a. There are 6 possible outcomes when rolling the first dice, for each outcome, rolling the second dice can result in 6 possible outcomes, thus there are $6\times 6=36$ outcomes in the sample space S .

b. When the sum of numbers appearing one the dice is equal to 7, it can be

$$7=1+6=6+1=2+5=5+2=3+4=4+3$$

Where $i+j$ denotes an outcome in the sample space: rolling the first dice gets i, rolling the second one gets j . Thus there are 6 possible outcomes when the sum of two dices is 7, the probability of interest is

$$\frac{\text{number of outcomes if sum is 7}}{\text{total number of equally likely outcomes}}=\frac{6}{36}=\frac{1}{6}$$

2.80

We know that

$$P(T)=\frac{4}{12}=\frac{1}{3},P(D)=\frac{6}{12}=\frac{1}{2}$$

The event that neither a three-point specialists nor a defensive standouts is selected can be denoted by $\overline{T}\cap\overline{D}$, the probability of this event is

$$P\left(\overline{T}\cap\overline{D}\right)=\frac{5}{12}$$

a. The event that the selected player is a three-point specialist and a defensive standout can be denoted by $T\cap D$, the probability of interest is

$$\begin{aligned}P\left(T\cap D\right)&=P(T)+P(D)-P(T\cup D)\\&=\frac{1}{3}+\frac{1}{2}-\left(1-P\left(\overline{T\cup D}\right)\right)\\&=\frac{5}{6}-\left(1-P\left(\overline{T}\cap\overline{D}\right)\right)\\&=\frac{5}{6}-\left(1-\frac{5}{12}\right)\\&=\frac{1}{4}\end{aligned}$$

b. The event that the selected player is a defensive standout but not a three-point specialist can be denoted by $\overline{T}\cap D$, the probability of interest is

$$P\left(\overline{T}\cap D\right)=P(D)-P(T\cap D)$$
$$=\frac{1}{2}-\frac{1}{4}$$
$$=\frac{1}{4}$$

c. The event that the selected player is either a three-point specialist or a defensive standout can be denoted by $T\cup D$, the probability of interest is

$$P\left(T\cup D\right)=1-P\left(\overline{T\cup D}\right)$$
$$=1-P\left(\overline{T}\cap\overline{D}\right)$$
$$=1-\frac{5}{12}$$
$$=\frac{7}{12}$$

3.17

Assuming that this village has the same number of sons and daughters, and probability that the eldest child is a boy and the probability that the eldest child is a girl are the same.

Let D denote the event that a child in the village is a daughter, E denote the event that a given child in the village is the eldest one. Then we have

$$P(D)=0.5, P(E)=0.5$$

Because D and E are independent, then

$$P(DE)=P(D)P(E)=0.25$$

The probability that a daughter is the eldest one is

$$P(E\mid D)=\frac{P(DE)}{P(D)}=\frac{0.25}{0.5}=0.5$$

3.34

Current can not flow from s to t only when all these 3 events happen: at least one of relay 1 and 2 close, relay 3 closes, relay 4 closes.

$$P(\text{at least one of relay 1 and 2 close})=1-P(\text{neither relay 1 nor relay 2 closes})$$
$$=1-(1-0.4)\cdot(1-0.6)$$
$$=0.76$$

$$P(\text{relay 3 closes})=0.5\,,\ P(\text{relay 4 closes})=0.9$$

Because the switches operate independently of each other, then

$$
\begin{aligned}
P(\text{current will flow from s to t}) &= 1 - P(\text{current will not flow from s to t}) \\
&= 1 - P(\text{at least one of relay 1 and 2 close}) \cdot P(\text{relay 3 closes}) \cdot P(\text{relay 4 closes}) \\
&= 1 - 0.76 \cdot 0.5 \cdot 0.9 \\
&= 0.658
\end{aligned}
$$

3.49

For a random ly selected household with a child under age 6, let A denote the event that the household is with female head, B denote the event that household is with male head, D de note the event that it is in poverty , M denote the event that it is with married-couple. Note that A, B, M form a partition of sample space.
We know that

$$
\begin{aligned}
&P(D \mid A) = 0.483, P(D \mid B) = 0.24, P(D \mid M) = 0.087 \\
&P(A) = 0.222, P(B) = 0.055, P(M) = 0.723
\end{aligned}
$$

a. Using the Theorem of Total Probability, we have

$$
\begin{aligned}
P(D) &= P(D \mid A) \cdot P(A) + P(D \mid B) \cdot P(B) + P(D \mid M) \cdot P(M) \\
&= 0.483 \cdot 0.222 + 0.24 \cdot 0.055 + 0.087 \cdot 0.723 \\
&= 0.183
\end{aligned}
$$

b. The probability of interest is

$$
P(M \mid D) = \frac{P(MD)}{P(D)} = \frac{P(D \mid M) \cdot P(M)}{P(D)} = \frac{0.087 \cdot 0.723}{0.183} = 0.344
$$

c. $P(M \mid D) \neq P(M)$, after we know that a household is in poverty, the probability that a household has married-couple changes, therefore, a household is in poverty or not is not independent of the type of the head of household.

3.64

These two persons can p erform a highly speci alized task in a business, im plying that they are und er age 65. Let A denote the even t that a given inoculated person catches flu, B denote the even t that a given person without vaccin ation catches flu, E denote the event that a given person is exposed to flu during the flu season. Then we know

$$
P(E) = 0.7, P(\overline{A} \mid E) = 0.8, P(B \mid E) = 0.9
$$

Because $A \subseteq E, B \subseteq E$,

$$
\begin{aligned}
P(A) &= P(E)P(A \mid E) = 0.7 \cdot (1 - 0.8) = 0.14 \\
P(B) &= P(E)P(B \mid E) = 0.7 \cdot 0.9 = 0.63
\end{aligned}
$$

These two persons are in dif ferent location and are not in contact with sam e people and each other, thus events A and B are independent. The probability of interest is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 0.14 + 0.63 - 0.14 \cdot 0.63 \\ &= 0.6818 \end{aligned}$$

3.76

The sample space is

$$S = \{\text{(head,head),(head, tail),(tail,head),(tail,tail)}\}$$

We know that $P(A) = 0.5, P(B) = 0.5$

If both tosses yield the sam e outcome, it can be (head, head) or (tail, ta il). And there are 4 outcomes in the sample space, thus $P(C) = \frac{2}{4} = 0.5$

When all of events A, B, C happen, it can only be (head, head), thus

$$P(ABC) = \frac{1}{4} = 0.25$$

We find that

$$P(A)P(B)P(C) = 0.125 \neq P(ABC)$$

Thus A, B and C are not independent.