4.5

We need to assume each outcome is equally likely for each person to choose each door. Each person has four choices, thus there are $4^3 = 64$ possible outcomes for three people to enter the building. If x people choose door I, there are $\binom{3}{x}$ ways to specify the x people, and for the remaining $3-x$ people, each of them has 3 choices to enter the building. Thus there are $\binom{3}{x} \cdot 3^{3-x}$ possible outcomes if x people choose entrance one, the probability of $X = x$ is

$$P(X = x) = \frac{\binom{3}{x} 3^{3-x}}{64}, \quad x = 0,1,2,3$$

Then $p(0) = \frac{27}{64}, p(1) = \frac{27}{64}, p(2) = \frac{9}{64}, p(3) = \frac{1}{64}$

4.6

(a) For a given structure fire, the probability that it is caused by cooking is

$$p = \frac{29706}{52006} = 0.5712$$

Now if x out of four are caused by cooking, there are $\binom{4}{x}$ ways to specify the x fires that are caused by cooking, and each outcome is equally likely, with probability $p^x(1-p)^{4-x}$, thus the probability of $X = x$ is

$$P(X = x) = \binom{4}{x} p^x (1-p)^{4-x}, \quad x = 0,1,2,3,4$$

Then we have

x	0	1	2	3	4
p(x)	0.0338	0.1801	0.3599	0.3197	0.1065

(b) The probability of interest is

$$P(X \geq 1) = 1 - P(x = 0) = 1 - 0.0338 = 0.9662$$

4.7

(a) X can take value 0,1,2,3,4,5,6. Let Y denote the number of sales in the first day, Z denotes the number of sales in the second day, Y and Z are independent. Then

$$P(X=0)=P(Y=0,Z=0)=P(Y=0)\cdot P(Z=0)=0.5\cdot 0.5=0.25$$

$$P(X=1)=P(Y=1)P(Z=0)+P(Y=0)P(Z=1)=0.3\cdot 0.5+0.5\cdot 0.3=0.3$$

$$\begin{aligned}P(X=2)&=P(Y=2)P(Z=0)+P(Y=0)P(Z=2)+P(Y=1)P(Z=1)\\&=0.15\cdot 0.5+0.5\cdot 0.15+0.3\cdot 0.3\\&=0.24\end{aligned}$$

$$\begin{aligned}P(X=3)&=P(Y=3)P(Z=0)+P(Y=0)P(Z=3)+P(Y=1)P(Z=2)+P(Y=2)P(Z=1)\\&=0.05\cdot 0.5+0.5\cdot 0.05+0.3\cdot 0.15+0.15\cdot 0.3\\&=0.14\end{aligned}$$

$$\begin{aligned}P(X=4)&=P(Y=1)P(Z=3)+P(Y=3)P(Z=1)+P(Y=2)P(Z=2)\\&=0.3\cdot 0.05+0.05\cdot 0.3+0.15\cdot 0.15\\&=0.0525\end{aligned}$$

$$\begin{aligned}P(X=5)&=P(Y=3)P(Z=2)+P(Y=2)P(Z=3)\\&=0.05\cdot 0.15+0.15\cdot 0.05\\&=0.015\end{aligned}$$

$$\begin{aligned}P(X=6)&=P(Y=3)P(Z=3)\\&=0.05\cdot 0.05\\&=0.0025\end{aligned}$$

(b) The probability of interest is

$$\begin{aligned}P(X\geq 2)&=1-P(X=0)-P(X=1)\\&=1-0.25-0.3\\&=0.45\end{aligned}$$

4.8

a. there are $\binom{4}{2}=6$ ways to choose two chips out of four in total.

$X=0$: only one possible outcome, we choose the two non-defective ones. Then

$$P(X=0)=\frac{1}{6}$$

$X=1$: we have chosen 1 defective and 1 non-defective, there are two ways to specify the defective one and there are two ways to specify the non-defective one, thus there are 4 possible outcomes. Then

$$P(X=1)=\frac{4}{6}=\frac{2}{3}$$

$X=2$: only one possible outcome, we choose the two defective ones. Then

$$P(X=2)=\frac{1}{6}$$

(b) The probability of interest is

$$P(X\le 1)=P(X=0)+P(X=1)=\frac{5}{6}$$

4.9

For a given people who enter a blood bank, the probability that he has type O+ blood is that

$$p=\frac{1}{3}$$

Now if x out of three people have type O+ blood, there are $\binom{3}{x}$ ways to specify the three people who have type O+ blood, and each outcome is equally likely, with probability $p^x(1-p)^{3-x}$, thus the probability of $X=x$ is

$$P(X=x)=\binom{3}{x}p^x(1-p)^{3-x},\quad x=0,1,2,3$$

Then we have

x	0	1	2	3
p(x)	8/27	4/9	2/9	1/27

Let $q=\frac{1}{20}$, in the same way we can derive that

$$P(Y=y)=\binom{3}{y}q^y(1-q)^{3-y},\quad x=0,1,2,3$$

Then we have

y	0	1	2	3
p(y)	0.857	0.135	0.007	0.000125

(b) For $X+Y$, $p=\frac{1}{3}+\frac{1}{20}=\frac{23}{60}$, in the same way we can derive

$$P(X+Y=u)=\binom{3}{u}p^u(1-p)^{3-u},\quad u=0,1,2,3$$

Then we have

u	0	1	2	3
p(u)	0.2345	0.4373	0.2718	0.0563

4.13

X can take value in $S=\{1,2,3,4,5,6,7\}$. For $k\in S$, the probability mass function is

$$p(k)=P(X=k)=P(\text{wrong key in the first } k\text{ -1 tries})\cdot P(\text{right key in the } k\text{th try})$$
$$=\prod_{i=1}^{k-1}\frac{7-i}{7-i+1}\cdot\frac{1}{7-k+1}$$
$$=\frac{7-k+1}{7}\cdot\frac{1}{7-k+1}$$
$$=\frac{1}{7}$$

4.14

There are $P_5^5=5!=120$ ways to put five letters in five envelopes, each outcome has the same probability. Let X denote the exact number of letters that are placed in the correct envelope. If $X=x\in S=\{0,1,2,3,5\}$

If $X=5$, all letters are in the correct envelope, there is only one possible outcome, thus

$$P(X=5)=\frac{1}{120}$$

If $X=3$, there are $\binom{5}{3}=10$ ways to specify the 3 letters that are placed in the right envelope. For the remaining 2 letters and corresponding envelopes, there is only one way to put in the wrong order, thus

$$P(X=3)=\frac{10\cdot 1}{120}=\frac{1}{12}$$

If $X=2$, there are $\binom{5}{2}=10$ ways to specify the 2 letters that are placed in the right envelope. For the remaining 3 letters and corresponding envelopes, there are only two ways to put all 3 in the wrong order, thus

$$P(X=2)=\frac{10\cdot 2}{120}=\frac{1}{6}$$

If $X=1$, there are $\binom{5}{1}=5$ ways to specify the letter that is placed in the right envelope.

For the remaining 4 letters and corresponding envelopes, there are 9 ways to put all 4 in the wrong order, thus

$$P(X=1)=\frac{5\cdot 9}{120}=\frac{3}{8}$$

Then we have

$$
\begin{aligned}
P(X=0) &= 1-P(X=1)-P(X=2)-P(X=3)-P(X=5) \\
&= 1-\frac{1}{120}-\frac{1}{12}-\frac{1}{6}-\frac{3}{8} \\
&= \frac{11}{30}
\end{aligned}
$$

4.22

Let X denote the number of red grouper caught by a given boat. The expect value is

$$E\left(X\right)=\sum_{x=0}^{2} xp(x)=0\cdot 0.2+1\cdot 0.7+2\cdot 0.1=0.9$$

Now,

$$E\left(X^2\right)=\sum_{x=0}^{2} x^2 p(x)=0^2\cdot 0.2+1^2\cdot 0.7+2^2\cdot 0.1=1.1$$

Then by theorem 4.3, the variance is

$$V\left(X\right)=E\left(X^2\right)-\left(EX\right)^2=1.1-0.9^2=0.29$$

The standard deviation is

$$\sigma=\sqrt{V(X)}=0.5385$$

4.24

a. Let X denote the number of contracts that are assigned to firm I. there are $3^2=9$ ways to assign two contracts to three companies.

If $X=2$, there is only one possible outcome, thus

$$P\left(X=2\right)=\frac{1}{9}$$

If $X=1$, there are two ways to specify which contract is assigned to firm I, and there are two ways to assign the other contract to firm II and firm III, thus

$$P\left(X=1\right)=\frac{2\cdot 2}{9}=\frac{4}{9}$$

Now for $X=0$,

$$P\left(X=0\right)=1-P(X=1)-P(X=2)=1-\frac{4}{9}-\frac{1}{9}=\frac{4}{9}$$

Let Y denote the potential profit for firm I, then $Y=90000X$, the expected potential profit is

$$
\begin{aligned}
E(Y) &= E(90000X) \\
&= 90000E(X) \\
&= 90000\left(2\cdot\frac{1}{9}+1\cdot\frac{4}{9}+0\cdot\frac{4}{9}\right) \\
&= 60000
\end{aligned}
$$

b. Let U denote the potential profit for firms I and II together, V denotes the potential profit for firm III. Then $U+V$ is the potential profit for the three companies together, thus

$$U+V=2\times 90000=180000$$

(Note: here the random variable U +V is a fixed value, it means $P(U+V=180000)=1$)

The expected value of $U+V$ is

$$E(U+V)=180000$$

From the result in (a), it is easy to know that $E(V)=60000$, then

$$E(U)=E(U+V)-E(V)=180000-60000=120000$$

4.27

a. Set $1-\frac{1}{k^2}=0.9$, then $k=\sqrt{10}=3.16$, then the interval

$$[\mu-k\sigma,\mu-k\sigma]=[2.472,7.528]$$

includes at least 90% of weekly figures for the number of breakdown.

b.

Let $k=3.75$, then

$$\begin{aligned} P(X>8) &\le P(|X-5|>3) \\ &= P(|X-\mu|>k\sigma) \\ &\le \frac{1}{k^2} \\ &= 0.071 \end{aligned}$$

Thus the director is safe in making this claim.

4.31

There are $26^3\cdot 10^4=175760000$ possible combinations of three letters and four numbers which form the sample space. Each outcome in the sample space has the same probability.

For the first lottery, there are $3!\cdot 4!=144$ outcomes in the sample space that have the same letters and numbers.

For the second lottery, there are $3\cdot 4!=72$ outcomes in the sample space that have the same letters and numbers.

For the third lottery, there are $3!\cdot 2\binom{4}{2}=72$ outcomes in the sample space that have the same letters and numbers.

For the fourth lottery, there are $3!\cdot 2\binom{4}{2}=72$ outcomes in the sample space that have the same letters and numbers.

Let W denote that whether you can win the lottery, that is, $W=1$ if you win, otherwise $W=0$. Then the probability of winning the grand prize is

$$P(W=1)=P(\text{winning the grand prize})=\frac{144+72+72+72}{175760000}=\frac{9}{4394000}$$

The expected profit is

$$E(W)=1000000\cdot P(W=1)=2.05$$

Thus the coupon is worth mailing back.

4.33

Let X denote the number of demand of these items on a given day. Suppose the merchant stock n items. Since the demand would not exceed four items, n should not be larger than 4. The minimum daily demand is two items, thus n should not be smaller than 2. Let Y denote the profit on a give day, then we have

$$Y=1.2\min\{X,n\}-n$$

We need to discuss the following situations:

i) $n=4$, then

$$Y=1.2X-4$$

The expected daily profit is

$$\begin{aligned}E(Y)&=E(1.2X-4)\\&=1.2E(X)-4\\&=-0.04\end{aligned}$$

ii) $n=3$,

$$\begin{aligned}&\text{when } X=2, Y=1.2\min\{2,3\}-3=-0.6\\&\text{when } X=3, Y=1.2\min\{3,3\}-3=0.6\\&\text{when } X=4, Y=1.2\min\{4,3\}-3=0.6\end{aligned}$$

Thus the expected daily profit is

$$E(Y)=(-0.6)\times 0.2+0.6\times 0.3+0.6\times 0.5=0.36$$

ii) $n=2$,

$$Y=1.2n-n=0.2n=0.4$$

Thus the expected daily profit is

$$E(Y)=0.4$$

Therefore, the merchant should stock two items to maximize the expected daily profit.