Solutions to Homework 5

5.5

a. $\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} ce^{-x/10}dx = -10\,ce^{-x/10}\Big|_0^{+\infty} = 10c = 1$

thus $c = \frac{1}{10}$.

b. if $b \le 0$, $F(b) = 0$, if $b > 0$, then

$$F(b) = \int_{-\infty}^{b} f(x)dx = \int_0^b \frac{1}{10}e^{-x/10}dx = -e^{-x/10}\Big|_0^b = 1 - e^{-b/10}$$

thus

$$F(b) = \begin{cases} 1 - e^{-b/10}, & b > 0 \\ 0, & b \le 0 \end{cases}$$

c. $P(X \ge 15) = 1 - P(X < 15) = 1 - P(X \le 15) = 1 - (1 - e^{-15/10}) = e^{-3/2}$ =0.2231

d. $P(X \ge 20 \mid X \ge 5) = \frac{P(X \ge 20)}{P(X \ge 5)} = \frac{e^{-2}}{e^{-1/2}} = e^{-3/2} = 0.2231$

5.6

$$\begin{aligned} P(X \ge 4) &= \int_4^{+\infty} f(x)dx \\ &= \int_4^6 \frac{3}{32}(x-2)(6-x)\,dx \\ &= 0.5 \end{aligned}$$

5.8

b. $P(X \ge 5) = 1 - P(X < 5) = 1 - P(X \le 5) = 1 - F(5) = \frac{5}{12}$

c.the probability density function of X is

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} 0, & x < 0 \text{ or } x > 10 \\ \frac{x}{20}, & 0 \le x \le 4 \\ \frac{10-x}{30}, & 4 \le x \le 10 \end{cases}$$

5.12

$P(X \ge 0.5) = \int_{0.5}^{1} 12x^2(1-x)\,dx = \left(4x^3 - 3x^4\right)\Big|_{0.5}^{1} = \frac{11}{16}$

Let Y denote the number of samples that have a proportion of impurities exceeding 0.5.

four samples are independently selected, thus Y can be modeled as binomial distribution with $n = 4,\ p = \frac{11}{16}$

a. $P(Y = 1) = \binom{4}{1} p(1-p)^3 = 0.0839$

b. $P(Y \ge 1) = 1 - P(Y = 0) = 1 - \binom{4}{0} p^0 (1-p)^4 = 1 - 0.0095 = 0.9905$

5.14

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{+\infty} x f(x)dx \\
&= \int_{71}^{73} x \cdot \frac{1}{2} dx \\
&= \frac{1}{4} x^2 \Big|_{71}^{73} \\
&= 72
\end{aligned}
$$

to compute σ, we first find $E(X^2)$

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x)dx \\
&= \int_{71}^{73} x^2 \cdot \frac{1}{2} dx \\
&= \frac{1}{6} x^3 \Big|_{71}^{73} \\
&= \frac{15553}{3}
\end{aligned}
$$

then

$$V(X) = E(X^2) - \left[E(X)\right]^2 = \frac{15553}{3} - 72^2 = \frac{1}{3}$$

$$\sigma = \sqrt{V(X)} = \frac{\sqrt{3}}{3}$$

5.17

a.

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{+\infty} x f(x)dx \\
&= \int_{0}^{4} x \cdot \frac{3}{64} x^2 \left(4 - x\right) dx \\
&= 2.4
\end{aligned}
$$

to compute $V(X)$ and σ, we first find $E(X^2)$

$$E(X^2)=\int_{-\infty}^{+\infty}x^2 f(x)dx$$
$$=\int_0^4 x^2\cdot\frac{3}{64}x^2\left(4-x\right)dx$$
$$=6.4$$

Then we have

$$V(X)=E(X^2)-\left[E(X)\right]^2=6.4-2.4^2=0.64$$

$$\sigma=\sqrt{V(X)}=0.8$$

b.we first specify

$$1-\frac{1}{k^2}=0.75$$

then

$$k=2$$

Thus, the interval $\mu-2\sigma$ to $\mu+2\sigma$ will contain at least 75% of the probability. this interval is given by

$$[\mu-2\sigma,\mu+2\sigma]=[0.8,4]$$

5.22

the probability density function of X is

$$f(x)=\frac{d}{dx}F(x)=\begin{cases}0, x<0 \text{ or } x>10\\ \frac{x}{20}, 0\le x\le 4\\ \frac{10-x}{30}, 4\le x\le 10\end{cases}$$

The mean number of minutes that Jerry is early for appointments is

$$E(X)=\int_{-\infty}^{+\infty}xf(x)dx$$
$$=\int_0^4 x\cdot\frac{x}{20}dx+\int_4^{10}x\cdot\frac{10-x}{30}dx$$
$$=\frac{14}{3}$$

5.27

a. $P(X>c)=\int_c^b\frac{1}{b-a}dx=\frac{b-c}{b-a}$

b. $P\left(X>d\mid X>c\right)=\frac{P\left(X>d\right)}{P\left(X>c\right)}=\frac{\frac{b-d}{b-a}}{\frac{b-c}{b-a}}=\frac{b-d}{b-c}$

5.39

Let X denote the travel distances, it is uniformly distributed between points a and b .

a. $P\left(X \le \frac{a+b}{2}\right) = \frac{\frac{a+b}{2} - a}{b-a} = 0.5$

b. $P\left(X - a > 3\left(b - X\right)\right) = P\left(X > \frac{a+3b}{4}\right) = 1 - F\left(\frac{a+3b}{4}\right) = 1 - \frac{\frac{a+3b}{4} - a}{b-a} = 0.25$

c. For any one of the three automobiles, the probability that this automobile travel past the midpoint between a and b is

$$P\left(X > \frac{a+b}{2}\right) = 1 - F\left(\frac{a+b}{2}\right) = 0.5$$

Let Y denote the number of automobiles that travel past the midpoint between a and b . Y can be modeled as a binomial distribution with $n = 3, p = 0.5$, then

$$P(Y = 1) = \binom{3}{1} 0.5\left(1 - 0.5\right)^2 = 0.375$$

5.59

Let X denote the service times at teller windows, it follows an exponential distribution with $\theta = 3.4$

a. $P(X > 2) = \int_2^\infty \frac{1}{3.4} e^{-\frac{\theta}{3.4}} d\theta = -e^{-\frac{\theta}{3.4}}\Big|_2^\infty = e^{-\frac{2}{3.4}} = 0.555$

b. $P\left(X > 4 \mid X > 2\right) = \frac{P\left(X > 4\right)}{P\left(X > 2\right)} = \frac{e^{-\frac{4}{3.4}}}{e^{-\frac{2}{3.4}}} = e^{-\frac{2}{3.4}} = 0.555$