Solutions to Homework 6

5.63
Solve equation

$$0.5 = F(m) = 1 - e^{-\frac{m}{\theta}}$$

We get

$$m = \theta \ln 2 = 0.693\theta < \theta$$

5.65
Let X denote the daily rainfall during a randomly selected September day.

a. $E(X) = \alpha\beta = 8, \quad V(X) = \alpha\beta^2 = 160$

b. $1 - \frac{1}{k^2} = 0.75$, then we have $k = 2$. From Tchebysheff's Theorem,

$$P(|X - \mu| \le 2\sigma) \ge 1 - \frac{1}{k^2} = 0.75$$

The interval $[\mu - 2\sigma, \mu + 2\sigma] = \left[8 - 8\sqrt{10}, 8 + 8\sqrt{10}\right]$ will include the daily rainfall for a randomly selected September day with a probability of at lease 0.75. the rainfall is nonnegative, thus the interval $\left[0, 8 + 8\sqrt{10}\right]$ will include the daily rainfall for a randomly selected September day with a probability of at lease 0.75

5.74
Let X denote the annual maximum river flows.

a. $E(X) = \alpha\beta = 240, \quad V(X) = \alpha\beta^2 = 36000$

b. $1 - \frac{1}{k^2} = 8/9$, then we have $k = 3$. From Tchebysheff's Theorem,

$$P(|X - \mu| \le 3\sigma) \ge 1 - \frac{1}{k^2} = 8/9$$

The interval $[\mu - 3\sigma, \mu + 3\sigma] = \left[240 - 180\sqrt{10}, 240 + 180\sqrt{10}\right]$ will include the maximum annual flow with a probability of at lease 8/9.

5.87
Let X denote the diameter of steel shafts.
a.

$$P(0.98 \le X \le 1.02) = P\left(\frac{0.98-1.005}{0.01} \le \frac{X-\mu}{\sigma} \le \frac{1.02-1.005}{0.01}\right)$$
$$= P(-2.5 \le Z \le 1.5)$$
$$= 0.4332 + 0.4938$$
$$= 0.927$$

Thus the percentage of output of this operation will fail to meet specifications is

$100(1-0.927)\% = 7.3\%$

b. $\mu = 1.00$ minimizes the fraction that fail to meet specifications

5.85

Let X denote the amount spent for maintenance and repairs in a certain company in the next week, then X has an approximately normal distribution with mean of 600 and standard deviation of 40. Hence, the probability that actual costs will exceed the budgeted amount is

$$P(X > 700) = P(\frac{X-600}{40} > \frac{700-600}{40})$$
$$= P(Z > 2.5)$$
$$= 0.0062$$

Z has a standard normal distribution.

5.93

Let X denote the resistance of the certain type of capacitor, X has a normal distribution with mean of 800 and standard deviation of 200, then

a.The proportion of capacitors which will meet this specification is

$$P(900 \le X \le 1000) = P\left(\frac{900-800}{200} \le \frac{X-800}{200} \le \frac{1000-800}{200}\right)$$
$$= P(0.5 \le Z \le 1)$$
$$= P(0 \le Z \le 1) - P(0 \le Z \le 0.5)$$
$$= 0.3413 - 0.1915$$
$$= 0.1498$$

b. Define events A: the first capacitor chosen satisfy the specifications

B: the second capacitor chosen satisfy the specificaions

Because the two capacitors are randomly chosen, then

$$P(A \cap B) = P(A)P(B) = 0.1498^2 = 0.02244$$

5.94

Let X denote the length of a spotted sea trout, X has a normal distribution with mean of 22 and standard deviation of 4.

a. The probability that a fisher man catches a spotted sea trout within the legal limits is

$$P(14 \le X \le 24) = P\left(\frac{14-22}{4} \le \frac{X-22}{4} \le \frac{24-22}{4}\right)$$
$$= P(-2 \le Z \le 0.5)$$
$$= P(0 \le Z \le 2) + P(0 \le Z \le 0.5)$$
$$= 0.4772 + 0.1915$$
$$= 0.6687$$

b. We need to find x such that $P(X > x) = 0.05$. Now

$$P(X > x) = P\left(\frac{X-22}{4} > \frac{x-22}{4}\right)$$
$$= P\left(Z > \frac{x-22}{4}\right)$$

From Table 4, we know that if

$$P(Z > z_0) = 0.05$$

then $z_0 = 1.645$. Thus,

$$\frac{x-22}{4} = 1.645$$

and

$$x = 28.58$$

c. Let Y denote the number of trout out side the legal limit caught by the fisherman before he catches his first legal spotted sea trout, then Y can be modeled as the geometric distribution with the parameter $p = P(14 \le X \le 24) = 0.6687$, then

$$P(Y = 3) = (1-p)^3 p = 0.0243$$

5.105

a.

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^4(1-x)^2 dx$$
$$= k\int_0^1 (x^4 - 2x^5 + x^6)dx$$
$$= k(\frac{1}{5}x^5 - \frac{1}{3}x^6 + \frac{1}{7}x^7)\Big|_0^1$$
$$= \frac{k}{105}$$

To make $f(x)$ a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = \frac{k}{105} = 1$$

Thus

$$k = 105$$

b.

X has the Beta Distribution with $\alpha = 5, \beta = 3$, then we have

$$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{5}{5+3} = \frac{5}{8}$$

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{5}{192}$$

5.107

Let X denote the proportion of the fetuses in a randomly selected litter responding, then X has a Beta Distribution with $\alpha = 3, \beta = 2$

a. the mean litter proportion of fetuses responding to this hazard is

$$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{3+2} = 0.6$$

b. the probability is

$$\begin{aligned} P(X \le 0.2) &= \int_{-\infty}^{0.2} f(x)dx \\ &= \int_{0}^{0.2} 12x^2(1-x)dx \\ &= 12\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^{0.2} \\ &= 0.0272 \end{aligned}$$

5.116

Let X denote the proportion of pure iron in certain ore samples. The probability density function of X is

$$f(x) = \frac{\Gamma(4)}{\Gamma(3)\Gamma(1)}x^2 = 3x^2, \quad \text{for } 0 < x < 1$$

a. The probability that one of these samples will have more than 50% pure iron is

$$P(X > 0.5) = \int_{0.5}^{1} 3x^2 dx = x^3 \Big|_{0.5}^{1} = 0.875$$

b. b. The probability that one of these samples will have less than 30% pure iron is

$$P(X < 0.3) = \int_{0}^{0.3} 3x^2 dx = x^3 \Big|_{0}^{0.3} = 0.027$$

The probability that two out of three samples will have less than 30% pure iron is

$$\binom{3}{2}(0.027)^2(1-0.027) = 0.0021$$