# Solutions to Homework 7

**5.140**

For gamma distribution with parameters $\alpha$ and $\beta$, the moment generating function is

$$\begin{aligned}
M(t) &= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x/(\frac{\beta}{1-\beta t})} dx \\
&= \frac{\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\Gamma(\alpha)\beta^\alpha} \\
&= (1-\beta t)^{-\alpha}
\end{aligned}$$

**6.1**

a.

Let $Z$ denote the number of contracts assigned to firm III, then

$$\begin{aligned}
P(X=0, Y=0) = P(X=0, Y=0, Z=2) &= P(\text{contracts 1 assigned to III}) \cdot P(\text{contracts 2 assigned to III}) \\
&= \frac{1}{3} \cdot \frac{1}{3} \\
&= \frac{1}{9}
\end{aligned}$$

And

$$\begin{aligned}
P(X=1, Y=0) = P(X=1, Y=0, Z=1) &= P(\text{contracts 1 assigned to I}) \cdot P(\text{contracts 2 assigned to III}) \\
&\quad + P(\text{contracts 1 assigned to III}) \cdot P(\text{contracts 2 assigned to I}) \\
&= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \\
&= \frac{2}{9}
\end{aligned}$$

Similarly, we can derive the result in the table below

| | X=0 | X=1 | X=2 |
|---|---|---|---|
| Y=0 | 1/9 | 2/9 | 1/9 |
| Y=1 | 2/9 | 2/9 | 0 |
| Y=2 | 1/9 | 0 | 0 |

b.

the marginal distribution for $Y$ is

$$P(Y=0)=\sum_{i=0}^{2}P(X=i,Y=0)=\frac{4}{9}$$

$$P(Y=1)=\sum_{i=0}^{2}P(X=i,Y=1)=\frac{4}{9}$$

$$P(Y=2)=\sum_{i=0}^{2}P(X=i,Y=2)=\frac{1}{9}$$

c.

$$P(X=1|Y=1)=\frac{P(X=1,Y=1)}{P(Y=1)}=\frac{\frac{2}{9}}{\frac{4}{9}}=0.5$$

**6.5**

b.

$$\begin{aligned}P(X\le 0.2,Y\le 0.4)&=\int_{-\infty}^{0.4}\int_{-\infty}^{0.2}f(x,y)dxdy\\&=\int_{0}^{0.4}\int_{0}^{0.2}1dxdy\\&=0.08\end{aligned}$$

c.

$$\begin{aligned}P(0.1\le X\le 0.3,Y>0.4)&=\int_{0.4}^{\infty}\int_{0.1}^{0.3}f(x,y)dxdy\\&=\int_{0.4}^{1}\int_{0.1}^{0.3}1dxdy\\&=0.12\end{aligned}$$

**6.8**

Note: the joint pdf given here is wrong, it does not integrate to 1, change 50 to 25.

a.

$$\begin{aligned}P\left(X<0.5,Y<0.2\right)&=\int_{-\infty}^{0.5}\int_{-\infty}^{0.2}f(x,y)dxdy\\&=\int_{0.4}^{0.5}\int_{0.1}^{0.2}25dxdy\\&=0.25\end{aligned}$$

b.

$$P\left(X<0.5, Y>0.2\right)=\int_{-\infty}^{0.5}\int_{0.2}^{\infty} f(x,y)dxdy$$
$$=\int_{0.4}^{0.5}\int_{0.2}^{0.3} 25dxdy$$
$$=0.25$$

c.

$$P\left(X<0.5\right)=P\left(X<0.5,-\infty<Y<\infty\right)$$
$$=\int_{-\infty}^{0.5}\int_{-\infty}^{\infty} f(x,y)dxdy$$
$$=\int_{0.4}^{0.5}\int_{0.1}^{0.3} 25dxdy$$
$$=0.5$$

d.

$$P\left(X>0.5 \mid Y>0.2\right)=\frac{P\left(X>0.5, Y>0.2\right)}{P\left(Y>0.2\right)}$$
$$=\frac{\int_{0.5}^{\infty}\int_{0.2}^{\infty} f(x,y)dxdy}{\int_{-\infty}^{\infty}\int_{0.2}^{\infty} f(x,y)dxdy}$$
$$=\frac{\int_{0.5}^{0.6}\int_{0.2}^{0.3} 25dxdy}{\int_{0.4}^{0.6}\int_{0.2}^{0.3} 25dxdy}$$
$$=0.5$$

**6.11**

a. for $0\le x\le 1$, we have

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy$$
$$=\int_{0}^{1} 1dy$$
$$=1$$

Otherwise, $f_X(x)=0$.

here $X$ ~ uniform[0,1]

b. for $0 \le y \le 1$, we have

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\
&= \int_0^1 1dx \\
&= 1
\end{aligned}
$$

Otherwise, $f_Y(y) = 0$ .

here $Y$ ~ uniform[0,1]

**6.21**

a.

$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \begin{cases} 1, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

c. from 6.11, we have

$$
f_X(x) = \begin{cases} 1, 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
$$

$$
f_Y(y) = \begin{cases} 1, 0 \le y \le 1 \\ 0, \text{otherwise} \end{cases}
$$

then

$$
f_X(x) f_Y(y) = \begin{cases} 1, 0 \le x \le 1, 0 \le y \le 1 \\ 0, \quad \text{otherwise} \end{cases}
$$

Thus

$$
f_X(x) f_Y(y) = f_{X,Y}(x,y)
$$

$X$ and $Y$ are independent.

d. when $Y = 0.4$, the conditional density function becomes

$$
f_{X|Y}(x \mid y = 0.4) = \begin{cases} 1, \ 0 \le x \le 1 \\ 0, \text{otherwise} \end{cases}
$$

Then

$$P(X \geq 0.8 \mid Y = 0.4) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y = 0.4)dx$$
$$= \int_{0.8}^{1} 1dx$$
$$= 0.2$$

**6.22**

The marginal pdf of $Y$ is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$
$$= \int_{0.4}^{0.6} 25dx$$
$$= 5, \qquad 0.1 \leq y \leq 0.3$$

The conditional pdf for $X \mid Y = 0.25$ is

$$f_{X|Y}(x \mid y = 0.25) = \frac{f(x, 0.25)}{f_Y(0.25)} = \begin{cases} 5, & 0.4 \leq x \leq 0.6 \\ 0, & \text{otherwise} \end{cases}$$

Thus the probability of interest is

$$P\left(X > 0.5 \mid Y = 0.25\right) = \int_{0.5}^{\infty} f_{X|Y}(x \mid y = 0.25)dx$$
$$= \int_{0.5}^{0.6} 5dx$$
$$= 0.5$$