Solutions to Homework 8

6.23

For a fixed $Y = y \in [0,1]$, we have $0 \le x \le 1-y$, then the marginal pdf of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_0^{1-y} 2dx \\ &= 2(1-y), \qquad 0 \le y \le 1 \end{aligned}$$

Then, the conditional probability density function for X given $Y = y$ is

$$f_{X|Y}(x \mid y = 0.25) = \frac{f_{X,Y}(x, 0.25)}{f_Y(0.25)} = \frac{2}{2(1-0.25)} = \frac{4}{3}, \quad 0 \le x \le 0.75$$

The probability that chemical I comprise more than half of the mixture if a fourth of the mixture is chemical II is

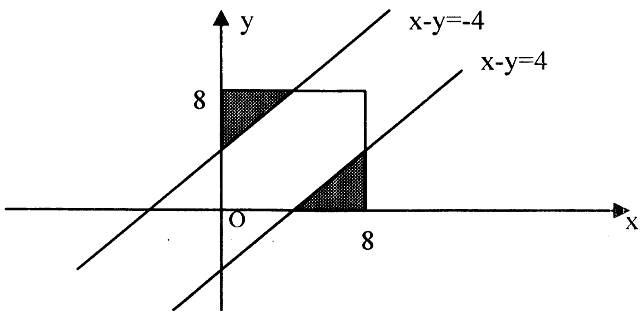
$$\begin{aligned} P(X > 0.5 \mid Y = 0.25) &= \int_{0.5}^{\infty} f_{X|Y}(x \mid y = 0.25)dx \\ &= \int_{0.5}^{0.75} \frac{4}{3}dx \\ &= \frac{1}{3} \end{aligned}$$

6.28

Let X denote the time of one inspector interrupting a production line in a given day. Let Y denote the time of the other inspector interrupting a production line in a given day. Then X and Y can both be modeled to have a uniform distribution in $[0,8]$.
Because X and Y are independent, the joint pdf of X and Y is

$$f(x,y) = \begin{cases} \frac{1}{64}, 0 \le x \le 8, 0 \le y \le 8 \\ 0, \text{ otherwise} \end{cases}$$

Define set $A = \{(x,y), 0 \le x \le 8, 0 \le y \le 8, |x-y| > 4\}$, A is the shadow area in the figure below



Thus the probability that the two interruptions will be more than four hours apart is

$$\begin{aligned}P(|X-Y|>4) &= \iint_A f(x,y)dxdy \\ &= \iint_A \frac{1}{64}dxdy \\ &= \frac{1}{64} \bullet S(\text{Shadow}) \\ &= \frac{1}{64}\cdot 2\cdot\frac{1}{2}\cdot 4^2 \\ &= \frac{1}{4}\end{aligned}$$

6.31

a.

$$\begin{aligned}E(X) &= 0\cdot P(X=0)+1\cdot P(X=1)+2\cdot P(X=2)+3\cdot P(X=3) \\ &= 1\cdot(0.04+0.05+0.07+0.10)+2\cdot(0.11+0.09+0.06+0.01)+3\cdot(0.10+0.06+0.02+0.01) \\ &= 1.37\end{aligned}$$

b.

$$\begin{aligned}E(Y) &= 0\cdot P(Y=0)+1\cdot P(Y=1)+2\cdot P(Y=2)+3\cdot P(Y=3) \\ &= 1\cdot 0.24+2\cdot 0.24+3\cdot 0.24 \\ &= 1.44\end{aligned}$$

c.

$$\begin{aligned}\text{cov}(X,Y) &= E(XY)-E(X)E(Y) \\ &= \sum_{x=0}^{3}\sum_{y=0}^{3} x\cdot y\cdot P(X=x,Y=y)-1.37\times 1.44 \\ &= -0.6128\end{aligned}$$

d.

$$\begin{aligned}V(X) &= E(X-E(X))^2 \\ &= 0.28(1.37-0)^2+0.26(1.37-1)^2+0.27(1.37-2)^2+0.19(1.37-3)^2 \\ &= 1.1731\end{aligned}$$

$$
\begin{aligned}
V(Y) &= E\left(Y-E(Y)\right)^2 \\
&= 0.28(1.44-0)^2+0.24(1.44-1)^2+0.24(1.44-2)^2+0.24(1.44-3)^2 \\
&= 1.2864
\end{aligned}
$$

Then the correlation is

$$\rho=\frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)V(Y)}}=\frac{-0.6128}{\sqrt{1.1731\times 1.2864}}=-0.4988$$

6.34

a. For fixed $X=x\in[0,2]$, we have $0\le y\le \frac{x}{2}$, then the marginal pdf of X is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_0^{x/2} 1dy \\
&= \frac{x}{2}, \quad 0\le x\le 2
\end{aligned}
$$

The mean of X is

$$
\begin{aligned}
E(X) &= \int_0^2 x\cdot\frac{x}{2}dx \\
&= \left.\frac{x^3}{6}\right|_0^2 \\
&= \frac{4}{3}
\end{aligned}
$$

The variance of X is

$$
\begin{aligned}
V(X) &= E(X^2)-E(X)^2 \\
&= \int_0^2 x^2\cdot\frac{x}{2}dx-\left(\frac{4}{3}\right)^2 \\
&= \left.\frac{x^4}{8}\right|_0^2-\left(\frac{4}{3}\right)^2 \\
&= \frac{2}{9}
\end{aligned}
$$

b. For fixed $Y=y\in[0,1]$, we have $2y\le x\le 2$, then the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$
$$= \int_{2y}^{2} 1dx$$
$$= 2-2y, \quad 0 \le y \le 1$$

The mean of Y is

$$E(Y) = \int_0^1 y(2-2y)dy$$
$$= \left(y^2 - \frac{2}{3}y^3 \right) \Bigg|_0^1$$
$$= \frac{1}{3}$$

The variance of Y is

$$V(Y) = E(Y^2) - E(Y)^2$$
$$= \int_0^1 y^2(2-y)dy - \left(\frac{1}{3}\right)^2$$
$$= \left(\frac{2}{3}y^3 - \frac{1}{4}y^4\right)\Bigg|_0^1 - \left(\frac{1}{3}\right)^2$$
$$= \frac{11}{36}$$

6.37

Define set $A = \{(x,y): 0 \le x \le 1, 0 \le y \le 1, 0 \le x+y \le 1\}$, for fixed $X = x$, $0 \le y \le 1-x$, then A can be equivalently written as

$$A = \{(x,y): 0 \le x \le 1, 0 \le y \le 1-x\},$$

Then we have

$$E(X) = \iint_A x f(x,y)dxdy$$
$$= \int_0^1 \int_0^{1-x} 2x dy dx$$
$$= \int_0^1 2x(1-x)dx$$
$$= \frac{1}{3}$$

$$E(Y) = \iint_A y f(x,y)dxdy$$

$$= \int_0^1 \int_0^{1-x} 2y dy dx$$

$$= \int_0^1 (1-x)^2 dx$$

$$= \frac{1}{3}$$

$$E(XY) = \iint_A xy f(x,y)dxdy$$

$$= \int_0^1 \int_0^{1-x} 2xy dy dx$$

$$= \int_0^1 x(1-x)^2 dx$$

$$= \frac{1}{12}$$

Thus the covariance between X and Y is

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{12} - \left(\frac{1}{3}\right)^2 = -\frac{1}{36}$$

b.

$$V(X) = \iint_A x^2 f(x,y)dxdy - E(X)^2$$

$$= \int_0^1 \int_0^{1-x} 2x^2 dy dx - \frac{1}{9}$$

$$= \int_0^1 2x^2(1-x)dx - \frac{1}{9}$$

$$= \frac{1}{18}$$

$$V(Y) = \iint_A y^2 f(x,y)dxdy - E(Y)^2$$

$$= \int_0^1 \int_0^{1-x} 2y^2 dy dx - \frac{1}{9}$$

$$= \int_0^1 \frac{2}{3}(1-x)^3 dx - \frac{1}{9}$$

$$= \frac{1}{18}$$

Then the correlation between X and Y is

$$\rho = \frac{\text{cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-\frac{1}{36}}{\sqrt{\frac{1}{18}\cdot\frac{1}{18}}} = -0.5$$

6.38

The expected value is

$$E(Z) = E(X+Y) = E(X) + E(Y) = 56 + 5 = 61$$

Because X and Y is independent, the variance is

$$V(Z) = V(X+Y) = V(X) + V(Y) = 16 + 4 = 20$$

6.40

a. Define set $A = \{(x,y): 0 \le y \le x < \infty, x - y > 1\}$, for fixed $Y = y$, $y+1 < x < \infty$, then A can be equivalently written as

$$A = \{(x,y): 0 \le y < \infty, y+1 < x < \infty\},$$

Then we have

$$\begin{aligned} P(X - Y > 1) &= \iint_A f(x,y)dxdy \\ &= \int_0^{\infty}\int_{y+1}^{\infty} e^{-x}dxdy \\ &= \int_0^{\infty} e^{-y-1}dy \\ &= e^{-1} \end{aligned}$$

b. Define set $B = \{(x,y): 0 \le y \le x < \infty\} = \{(x,y): 0 \le y < \infty, y \le x < \infty\}$, then

$$\begin{aligned} E(X - Y) &= \iint_B (x-y)f(x,y)dxdy \\ &= \int_0^{\infty}\int_y^{\infty} (x-y)e^{-x}dxdy \\ &= \int_0^{\infty} e^{-y}dy \\ &= 1 \end{aligned}$$

c.

$$
\begin{aligned}
V(X-Y) &= \iint_B (x-y)^2 f(x,y)dxdy - \left[E(X-Y)\right]^2 \\
&= \int_0^{\infty}\int_y^{\infty}(x-y)^2 e^{-x}dxdy - 1 \\
&= \int_0^{\infty} 2e^{-y}dy - 1 \\
&= 2-1 \\
&= 1
\end{aligned}
$$

Thus the standard deviation of the time spent at the sevice window is 1.

d.

Define set $C=\{(x,y):0\le y\le x<\infty, x-y>2\}$, for fixed $Y=y$, $y+2<x<\infty$, then C can be equivalently written as

$$C=\{(x,y):0\le y<\infty, y+2<x<\infty\},$$

Then we have

$$
\begin{aligned}
P(X-Y>2) &= \iint_C f(x,y)dxdy \\
&= \int_0^{\infty}\int_{y+2}^{\infty} e^{-x}dxdy \\
&= \int_0^{\infty} e^{-y-2}dy \\
&= e^{-2} \\
&= 0.1353
\end{aligned}
$$

Thus it is not highly likely that a customer will spend more than 2 minutes at the service window.

6.49

For a fixed $x\in[0,1]$, we have $0\le y\le 1-x$, then the marginal pdf of X is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_0^{1-x} 2dy \\
&= 2(1-x), \qquad 0\le x\le 1
\end{aligned}
$$

Then, the conditional probability density function for Y given $X=0.25$ is

$$f_{Y|X}(y\mid x=0.25)=\frac{f_{X,Y}(x,0.25)}{f_X(0.25)}=\frac{2}{2(1-0.25)}=\frac{4}{3},\quad 0\le y\le 0.75$$

a.

$$P(Y < 0.25 \mid X = 0.25) = \int_0^{0.25} f_{Y|X}(y \mid x = 0.25)dy$$

$$= \int_0^{0.25} \frac{4}{3} dy$$

$$= \frac{1}{3}$$

b. the mean proportion of chemical Y in the insecticide if 25% of the insecticide is chemical X is

$$E[Y \mid X = 0.25] = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x = 0.25)dy$$

$$= \int_0^{0.75} \frac{4}{3} y dy$$

$$= \frac{3}{8}$$

c. the variance is

$$V[Y \mid X = 0.25] = E[Y^2 \mid X = 0.25] - \{E[Y \mid X = 0.25]\}^2$$

$$= \int_{-\infty}^{\infty} y^2 f_{Y|X}(y \mid x = 0.25)dy - \left(\frac{3}{8}\right)^2$$

$$= \int_0^{0.75} \frac{4}{3} y^2 dy - \left(\frac{3}{8}\right)^2$$

$$= \frac{3}{64}$$

Then the standard deviation is $\sqrt{\frac{3}{64}} = 0.2165$

6.56 (NOT INCLUDED IN THE SYLLABUS)

Let P denote the probability of observing a defective item, we know that $P \sim uniform(0, \frac{1}{4}), X \mid P \sim \text{binomial}(n, p)$

a. the joint pdf of X and P is

$$f_{X,P}(x,p) = f_{X|P}(x \mid p) \cdot f_P(p)$$

$$= 4\binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,\ldots,n \quad 0 < p < \frac{1}{4}$$

b. the marginal distribution of X is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,P}(x,p)\,dp \\
&= \int_0^{1/4} 4\binom{n}{x} p^x (1-p)^{n-x}\,dp \\
&= 4\binom{n}{x} \int_0^{1/4} \sum_{k=0}^{n-x} (-1)^k \binom{n-x}{k} p^{k+x}\,dp \\
&= 4\binom{n}{x} \sum_{k=0}^{n-x} (-1)^k \binom{n-x}{k} \frac{\left(\frac{1}{4}\right)^{k+x+1}}{k+x+1}
\end{aligned}
$$