SOLUTIONS TO HOMEWORK 9

6.26 (a) $P(X<2, Y>1) = P((X,Y)\in A)$,

where $A = \{(x,y): x<2, y\geq 1\}$.

Range of density $f_{X,Y}$ is denoted by R.

$R = \{(x,y): 0 \leq y \leq x < \infty\}$

$$\Rightarrow A\cap R = \{(x,y): x<2, y>1, 0\leq y\leq x<\infty\}$$
$$= \{(x,y): 1\leq x\leq 2, \ 1\leq y\leq x\}$$

Hence,

$$P((X,Y)\in A) = \iint_{A\cap R} f_{X,Y}(x,y)\, dy\, dx$$
$$= \int_1^2\int_1^x e^{-x}\, dy\, dx$$
$$= \int_1^2 (x-1)e^{-x}\, dx$$
$$= \int_1^2 xe^{-x}\, dx - \int_1^2 e^{-x}\, dx$$
$$= \left[-xe^{-x}\right]_1^2 + \int_1^2 e^{-x}\, dx - \int_1^2 e^{-x}\, dx$$
$$= e^{-1} - 2e^{-2}$$

(b) $P(X\geq 2Y) = P((X,Y)\in A)$, where $A = \{(x,y): x\geq 2y)$

$\Rightarrow A\cap R = \{(x,y): x\geq 2y, \ 0\leq y\leq x<\infty\}$

$$= \left\{ (x,y): \quad 0 \le x \le \infty, \quad 0 \le y \le \frac{x}{2} \right\}$$

Hence, $P\left((X,Y) \in A \right)$

$$= \iint_{A \cap R} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^\infty \int_0^{x/2} e^{-x}\, dy\, dx$$

$$= \int_0^\infty \frac{x}{2} e^{-x}\, dx$$

$$= \frac{\Gamma(2)}{2}$$

$$= \frac{1}{2}.$$

$\boxed{(c)}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \begin{cases} \int_0^x e^{-x}\, dy & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

$$= \begin{cases} x e^{-x} & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

$\boxed{(d)}$ $$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Note that $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

$$= \begin{cases} \int_y^{\infty} e^{-x}\,dx & \text{if } y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-y} & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $f_{X|Y=y}(x) = \begin{cases} \dfrac{e^{-x}}{e^{-y}} & \text{if } y \leq x \leq \infty, \\ 0 & \text{otherwise} \end{cases}$

$$= \begin{cases} e^{-(x-y)} & \text{if } y \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

[redacted]

6.68

a. For the next five cars entering this intersections, let X_1 denote the number of cars that turn left, X_2 denote the number of cars that turn right, X_3 denote the number of cars that continue straight ahead, then X_1, X_2, X_3 has the multinomial distribution with $p_1 = 0.4, p_2 = 0.25, p_3 = 0.35$, then the probability of interest is

$$P(X_1 = 1, X_2 = 1, X_3 = 3) = \frac{5!}{1!1!3!} 0.4 \cdot 0.25 \cdot 0.35^3 = 0.08575$$

b. X_2 has the binomial distribution with $n = 5, p_2 = 0.25$,then

$$\begin{aligned} P(X_2 \geq 1) &= 1 - P(X_2 = 0) \\ &= 1 - \binom{5}{0} 0.25^0 (1 - 0.25)^5 \\ &= 0.7627 \end{aligned}$$

c. For the 100 cars entering this intersections, let Y denote the number of cars that turn left, Y has the binomial distribution with $p_1 = 0.4$, then

$$\begin{aligned} E(Y) &= np_1 = 40 \\ V(Y) &= np_1(1 - p_1) = 24 \end{aligned}$$

The assumption is that the trials are independent.

6.70

a. Let $X_1 \sim N(\mu_1, \sigma_1^2),\ X_2 \sim N(\mu_2, \sigma_2^2)$, then the moment generating function of $Y = aX_1 + bX_2$ is

$$\begin{aligned} M_Y(t) &= E\left(e^{tY}\right) \\ &= E\left(e^{taX_1} e^{tbX_2}\right) \\ &= E\left(e^{taX_1}\right) E\left(e^{tbX_2}\right) \\ &= M_{X_1}(at) M_{X_2}(bt) \\ &= e^{\mu_1 at + \sigma_1^2 a^2 t^2 / 2} e^{\mu_2 bt + \sigma_2^2 b^2 t^2 / 2} \\ &= e^{(a\mu_1 + b\mu_2)t + \left(a^2\sigma_1^2 + b^2\sigma_2^2\right)t^2 / 2} \end{aligned}$$

This is the moment-generating function of a normal distribution, thus $Y \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2})$

b.

$$E(Y) = \frac{d}{dt} M_Y(t)\big|_{t=0} = \left[\left(a\mu_1 + b\mu_2\right) + \left(a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2}\right)t\right] e^{(a\mu_1+b\mu_2)t + \left(a^2\sigma_1^{\,2}+b^2\sigma_2^{\,2}\right)t^2/2}\Big|_{t=0}$$
$$= a\mu_1 + b\mu_2$$

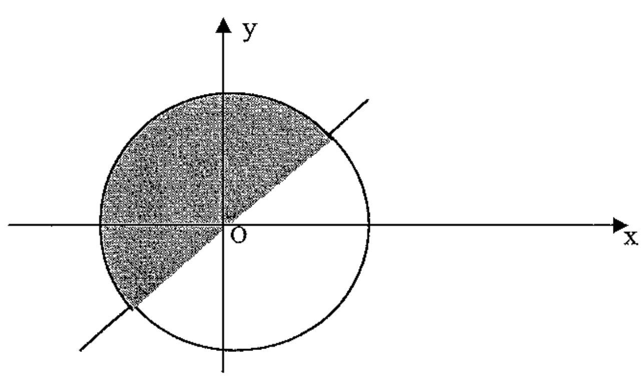
$$V(Y) = \frac{d^2}{dt^2} M_Y(t)\big|_{t=0} = \left[\left(a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2}\right) + \left(\left(a\mu_1 + b\mu_2\right) + \left(a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2}\right)t\right)^2\right] e^{(a\mu_1+b\mu_2)t + \left(a^2\sigma_1^{\,2}+b^2\sigma_2^{\,2}\right)t^2/2}\Big|_{t=0}$$
$$= a^2\sigma_1^{\,2} + b^2\sigma_2^{\,2}$$

6.86

a. for a fixed $X = x \in [-1,1]$, we have $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, the marginal density function of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-x^2}, \quad -1 \le x \le 1 \end{aligned}$$

b. define the set $A = \left\{(x,y) : x^2 + y^2 \le 1, x \le y\right\}$, then A is the shadow area in the figure below.



The probability of interest is

$$
\begin{aligned}
P(X \le Y) &= \iint_A f(x,y)\,dxdy \\
&= \frac{1}{\pi} S(A) \\
&= \frac{1}{\pi} \cdot \frac{\pi}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

7.3

a. X have the negative binomial distribution with $r = 4, p = 0.4$, then the distribution of X is

$$
\begin{aligned}
P(X = x) &= \binom{x+r-1}{r-1} p^r (1-p)^x \\
&= \binom{x+3}{3} 0.4^4 \cdot 0.6^x, \quad x = 0,1,2,\ldots
\end{aligned}
$$

b. $Y = X + 4$

c. the distribution of Y is

$$
\begin{aligned}
P(Y = y) &= P(X + 4 = y) \\
&= P(X = y - 4) \\
&= \binom{y-4+3}{3} 0.4^4 \cdot 0.6^{y-4} \\
&= \binom{y-1}{3} 0.4^4 \cdot 0.6^{y-4}, \quad y = 4,5,6,\ldots
\end{aligned}
$$

7.9

For $y \in (1, e)$,

$$
\begin{aligned}
F_Y(y) &= P(Y < y) \\
&= P(e^X < y) \\
&= P(X < \ln y) \\
&= \int_0^{\ln y} 1 dx \\
&= \ln y
\end{aligned}
$$

When $y \le 1$, $F_Y(y) = 0$, when $y \ge e$, $F_Y(y) = 1$

Thus the probability density function of Y is

$$f_Y(y) = \frac{d}{dx} F_Y(y) = \begin{cases} \frac{1}{y}, 1 < y < e \\ 0, \text{otherwise} \end{cases}$$

7.11

For $y \geq 0$,

$$\begin{aligned} F_Y(y) &= P(Y < y) \\ &= P(cX < y) \\ &= P(X < \frac{1}{c} y) \\ &= \int_0^{\frac{1}{c}y} \frac{1}{\theta} e^{-\frac{1}{\theta}x} dx \\ &= 1 - e^{-\frac{1}{c\theta}y} \end{aligned}$$

Then the probability density function of Y is

$$f_Y(y) = \frac{d}{dx} F_Y(y) = \begin{cases} \frac{1}{c\theta} e^{-\frac{1}{c\theta}y}, y \geq 0 \\ 0, \text{otherwise} \end{cases}$$

Thus $Y \sim$ exponential$(c\theta)$

7.13

For $y > 0$,

$$\begin{aligned} F_Y(y) &= P(Y < y) \\ &= P(e^X < y) \\ &= P(X < \ln y) \\ &= F_X(\ln y) \end{aligned}$$

Differentiation this equation, we have

$$\begin{aligned} f_Y(y) &= \frac{1}{y} f_X(\ln y) \\ &= \frac{1}{y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}, \quad y > 0 \end{aligned}$$

Otherwise, $f_Y(y) = 0$

7.20 $Y = e^X = g(X)$ $f_X(x) = \begin{cases} 1 & \text{if } x \in (0,1), \\ 0 & \text{otherwise} \end{cases}$

Step 1: $(\alpha, \beta) = (e^0, e^1) = (1, e)$.

Step 2: $Y = e^X \Rightarrow X = \log Y = h(Y)$.

Step 3: $h'(y) = \frac{1}{y} \neq 0$ if $y \in (1, e)$.

Step 4: $f_Y(y) = \begin{cases} f_X(h(y))\,|h'(y)| & \text{if } y \in (1,e), \\ 0 & \text{otherwise} \end{cases}$

$$= \begin{cases} \frac{1}{y} & \text{if } y \in (1,e), \\ 0 & \text{otherwise} \end{cases}$$

7.27 $f_{X_1,X_2}(x_1,x_2) = \begin{cases} 2 & \text{if } 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_1 + x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$

$\boxed{Y = X_1 + X_2}$ $f_{X_2}(x_2) = \begin{cases} \int_0^{1-x_2} 2\,dx_1 & \text{if } 0 \le x_2 \le 1, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2(1-x_2) & \text{if } 0 \le x_2 \le 1, \\ 0 & \text{otherwise} \end{cases}$

$$f_{X_1|X_2=x_2}(x_1) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

$$= \begin{cases} \frac{2}{2(1-x_2)} & \text{if } 0 \le x_1 \le 1 - x_2, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{1-x_2} & \text{if } 0 \le x_1 \le 1-x_2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $X_2 = x_2$, then $Y = X_1 + x_2$. We will now find $f_{Y|X_2=x_2}$ by the method of transformations.

$Y = g(X_1) = X_1 + x_2$

Step 1: $(\alpha, \beta) = (0 + x_2, 1 - x_2 + x_2) = (x_2, 1)$

Step 2: $X_1 = Y - x_2 = h(Y)$

Step 3: $\frac{d}{dy} h(y) = 1 \neq 0$ for $y \in (x_2, 1)$

Step 4:

$$f_{Y|X_2=x_2}(y) = \begin{cases} f_{X_1}(h(y)) \left|\frac{d}{dy} h(y)\right| & \text{if } y \in (x_2, 1) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{1-x_2} & \text{if } y \in (x_2, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X_2=x_2}(y)\, f_{X_2}(x_2)\, dx_2$$

Note that $f_{Y|X_2=x_2}(y) = 0$ if $y < x_2$ or $y > 1$.

Hence, if $0 \le y \le 1$,

$$f_Y(y) = \int_0^y \frac{1}{1-x_2} \, 2(1-x_2) \, dx_2$$

$$= \int_0^y 2 \, dx_2$$

$$= 2y.$$

Clearly if $y < 0$ or $y > 1$,

$$f_Y(y) = 0.$$

Hence $f_Y(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$

~~6.26 (a) $P(X < 2, Y > 1) = P((X,Y) \in A)$,~~

~~where $A = \{(x,y) : x < 2, y > 1\}$~~

~~$R =$ Range of density $f_{X,Y} = \{(x,y) : 0 \le y \le x < \infty\}$~~

~~$A \cap R = \{(x,y) : x < 2, y > 1, 0 \le y \le x < \infty\}$~~

~~$= \{(x,y) : 1 \le x \le 2, 1 \le y \le x\}$~~

~~$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\, dx\, dy = \int_1^2 \int_1^x e^{-x} \, dy\, dx$~~