LECTURE 11

Agenda:

1. Tchebysheff's theorem (from Lecture 10)
2. Bernoulli random variables

TCHEBYSHEFF'S THEOREM (CONTINUED)

Tchebysheff's theorem is typically quite conservative, but thats the price to pay for a general result.

__Example:__ Let X = # of heads in 3 tosses of a fair coin (independently)

Then, $P(X=0) = \frac{1}{8}$, $P(X=1) = \frac{3}{8}$, $P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$.

$E(X) = \frac{3}{2}$, $V(X) = \frac{3}{4}$.

Applying Tchebysheff's theorem with $k = 2$, we get that,

$$P\left(\left|X - \tfrac{3}{2}\right| < \sqrt{3}\right) \geq 1 - \frac{1}{2^2}$$

i.e $P(-0.232 < X < 3.232) \geq 0.75$

But this is very conservative.

WARNING: For a discrete random variable X,

$$E(X) = \sum_{x \in \mathcal{X}} x\, p_X(x) = \sum_{x \in \mathcal{X}} x\, P(X = x) \text{ is}$$

well-defined only if $\sum_{x \in \mathcal{X}} |x|\, p_X(x) < \infty$.

If $\mathcal{X}$ is a finite space, this condition will always hold. However, when $\mathcal{X}$ is not a finite space, one should always make sure that the expectations are finite.

<u>Example:</u> Suppose X is a random variable with range $\mathcal{X} = \{1, 2, 3, \text{------------}\}$, and the probability mass function of X is given by

$$P(X = x) = \frac{6}{\pi^2 x^2} \quad \text{for every } x \in \mathcal{X}.$$

(Note that $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, hence the above assignment of probabilities is valid.)

Then,

$$\sum_{x \in \mathcal{X}} |x|\, p_X(x) = \sum_{x=1}^{\infty} \frac{1}{x} = \infty.$$

Hence $E(X)$ is not well-defined in this case.

BERNOULLI RANDOM VARIABLES

An experiment with two possible outcomes is called a "Bernoulli experiment." or a "Bernoulli trial". Suppose one outcome of a Bernoulli trial is identified as success and the other outcome is identified as failure. Define the random variable X such that

$$X = \begin{cases} 1 & \text{if the outcome of the trial is a success,} \\ 0 & \text{otherwise.} \end{cases}$$

Let p denote the probability of success in the experiment, i.e,

$$P(X = 0) = 1 - p, \quad P(X = 1) = p.$$

Such a random variable is said to be a "Bernoulli random variable". The only parameter needed to describe the probability distribution of this random variable is p.

$$E(X) = 0 \cdot (1-p) + 1 \cdot p = p.$$

$$\begin{aligned} V(X) &= E(X^2) - (E(X))^2 \\ &= 0 \cdot (1-p) + 1 \cdot p - p^2 \\ &= p(1-p). \end{aligned}$$